Semantics of Sequent Calculi with Basic Structural Rules: Fuzziness Versus Non-Multiplicativity

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SEMANTICS OF SEQUENT CALCULI WITH BASIC STRUCTURAL RULES: FUZZINESS VERSUS NON-MULTIPLICATIVITY

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ABSTRACT. The main *general* result of the paper is that *basic* structural rules — Enlargement, Permutation and Contraction — (as well as Sharings) [and Cuts] are derivable in a {multiplicative} propositional two-side sequent calculus iff there is a class of {crisp} (reflexive) [transitive distributive] fuzzy two-side matrices such that any rule is derivable in the calculus iff it is true in the class, the "{}"/"()[]"-optional case being due to [15]/[13]. Likewise, fyzzyfying the notion of signed matrix [15], we extend the main result obtained therein beyond multiplicative calculi. As a representative application, we prove that the sequent calculus $\mathbb{LK}_{[\mathrm{S/C}]}$ resulted from Gentzen's LK [3] by adding the rules inverse to the logical ones and retaining as structural ones merely basic ones [and Sharing/Cut] is equivalent (in the sense of [9]) to the bounded version of Belnap's four-valued logic (cf. [2]) [resp., the *logic of paradox* [6]/ Kleene's three-valued logic [4]]. As a consequence of this equivalence, appropriate generic results of [9] concerning extensions of equivalent calculi and the advanced auxiliary results on extensions of the bounded versions of Kleene's three-valued logic and the logic of paradox proved here with using the generic algebraic tools elaborated in [12], we then prove that extensions of the Sharing/Cut-free version $\mathbb{LK}_{\mathrm{C/S}}$ of LK form a three/four-element chain/, consistent ones having same derivable sequents that provides a new profound insight into Cut Elimination in LK appearing to be just a consequence of the well-known regularity of operations of Belnap's four-valued logic. Likewise, by the mentioned regularity, we prove that the logic of paradox is the only proper consistent axiomatic extension of Belnap's four-valued logic. As a consequence, we conclude that $\mathbb{LK}_{\mathrm{S}}$ is the only proper consistent axiomatic extension of $\mathbb{LK}$.

1. INTRODUCTION

Most universal, natural and immediate semantics of sequent calculi (of miscellaneous kinds) arises from the fundamental study [9] treating such calculi as unversal Horn theories, algebraic systems [5] becoming model structures of sequent calculi. However, model structures of such a kind, being normally of infinite first-order signature, are too cumbersome. On the other hand, the main peculiarity of sequent calculi consists in possessing structural rules (of miscellaneous kinds). It has been properly taken into account in further studies [13], [14], [15] and [17], providing a perfectly finitary semantics of sequent calculi with structural rules. The present study advances this research paradigm in the following two principal respects. First, admitting *two-side* matrices with independent left/right truth predicates for evaluating formulas in the left/right sides of sequents, we extend [13] to sequent calculi with *basic* structural rules — Enlargement, Permutation and Contraction — as well

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as, *optionally*, Cut and/or Sharing. Second, fuzzifying the conception of *signed* matrix [15], we extend the main result of the mentioned study to non-multiplicative signed sequent calculi.

Perhaps, the most representative and illustrative instance to be explored within the present enhancement would be LK [3] without Cut and/or Sharing.

In this connection, recall that an equivalence (in the sense of [9]) between the Cut-free fragmentary version of Gentzen's LK [3] supplemented by the inverse logical rules and void of the empty sequent has been discovered and explored in [18]. The exemplifying part of the present study enhances the mentioned elaboration by adding truth and falsehood constants as primary connectives, proper incorporating the empty sequent (that appears as the conclusion of a Cut instance and, for this reason, should not be excluded, as it was made in [18]) and, what is main, involving Sharing-free versions of LK.

2. General underlying issues

Concerning algebras, propositional languages and logics and logical matrices, we entirely follow standard conventions adopted in [12] but using rather Fraktur than Calligraphic letters for denoting algebras, Calligraphic letters (possibly, with indices) being reserved for denoting logical matrices, their underlying algebras — viz., algebra reducts — being denoted by corresponding Fraktur letters (with same indices, if any). For other issues concerning Lattice Theory, Universal Algebra and Model Theory, specified explicitly neither therein nor here, the reader is referred to standard mathematical handbooks like [1], [5] and [20].

Below, we just specify certain particular set- and lattice-theoretical notations used here other than the standard ones like dom, img and π_i as well as some basic issues concerning axiomatic extensions of logics.

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω.

Likewise, as usual, functions are viewed as binary relations.

Given a set S [and any $K \subseteq \omega$], the set of all subsets of S[of cardinality $\in K$] is denoted by $\wp_{[K]}(S)$. Next, S-tuples (viz., functions with domain S) are often written in either vector $\vec{t}$ or sequence $\bar{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as t_s in that case. Given a one more set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s\in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s\in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a, a\rangle | a \in S\}$, relations of such a kind being referred to as *diagonal*, and $S^{*/+} \triangleq \bigcup_{i\in(\omega/(\omega\setminus 1))} S^i$. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = \operatorname{dom} \vec{a}$ of any $\vec{a} \in S^+$, put:

$$\diamond\vec{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\vec{a}\restriction(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Finally, given any $T \subseteq S$, we have the *characteristic function* $\chi_S^T \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\}))$ *of* T *in* S.

2.2. **Lattice-theoretic background.** *[Bounded] lattices* [1] are supposed to be of the signature $\Sigma_{+[,01]} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}])$, where $\wedge$ (conjunction) and $\vee$ (disjunction) are binary [while both $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively) are nullary].

Given any $n \in (\omega \setminus 2)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice given by the chain poset n ordered by ordinal inclusion.

Given any $\Sigma \supseteq \Sigma_{+[,01]}$ and any Σ-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\upharpoonright}\Sigma_{+[,01]}$ is a [bounded] lattice, the partial ordering of the latter is denoted by $\leqslant^{\mathfrak{A}}$, the lattice meet/join of any $X \in \wp_{(\omega\setminus 1)[\cup 1]}(A)$ being denoted by $(\bigwedge / \bigvee)^{\mathfrak{A}} X$, respectively.

2.3. **Axiomatic extensions of propositional logics.** Given any $\mathcal{A} \subseteq \mathrm{Fm}_\Sigma$, set $\mathrm{SI}_\Sigma(\mathcal{A}) \triangleq \{\sigma(\varphi) \mid \varphi \in \mathcal{A}, \sigma \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{Fm}_\Sigma)\}$ and $\mathrm{Mod}(\mathcal{A})$ the class of all Σ-matrices satisfying every element of $\mathcal{A}$.

An extension $\mathcal{L}'$ of a Σ-logic $\mathcal{L}$ is said to be *axiomatic*, whenever it is relatively axiomatized by a set $\mathcal{A}$ of Σ-axioms, in which case:

$$((\Gamma \vdash \varphi) \in \mathcal{L}') \Leftrightarrow (((\mathrm{SI}_\Sigma(\mathcal{A}) \cup \Gamma) \vdash \varphi) \in \mathcal{L}), \tag{2.1}$$

for all $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}_\Sigma$.

A Σ-matrix $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$ is said to be *consistent/truth-non-empty*, provided $D^{\mathcal{A}} \neq (A/\varnothing)$. The class of all [consistent] submatrices of members of a class of Σ-matrices M is denoted by $\mathbf{S}_{[*]}(\mathsf{M})$.

Proposition 2.1. *Let* M *be a class of* Σ*-matrices and* $\mathcal{A} \subseteq \mathrm{Fm}_\Sigma$*. Then, the axiomatic extension of the logic of* M *relatively axiomatized by* $\mathcal{A}$ *is the logic of* $\mathsf{S} \triangleq (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A}))$.

Proof. We use (2.1) tacitly. Consider any $(\Gamma \cup \{\varphi\}) \subseteq \mathrm{Fm}_\Sigma$.

First, assume $(\Gamma \cup \mathrm{SI}_\Sigma(\mathcal{A})) \vdash \varphi$ is satisfied in each member of M. Consider any $\mathcal{A} \in \mathsf{S}$ and any $h \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{A})$ such that $\Gamma \subseteq h^{-1}[D^{\mathcal{A}}]$, in which case there is some $\mathcal{B} \in \mathsf{M}$ such that $\mathcal{A}$ is a submatrix of $\mathcal{B}$, and so $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{B})$ and $\Gamma \subseteq h^{-1}[D^{\mathcal{B}}]$. Moreover, for every Σ-substitution σ, $(h \circ \sigma) \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{A})$, in which case $\mathrm{SI}_\Sigma(\mathcal{A}) \subseteq h^{-1}[D^{\mathcal{A}}] \subseteq h^{-1}[D^{\mathcal{B}}]$, and so $\varphi \in h^{-1}[(\mathrm{img}\, h) \cap D^{\mathcal{B}}] \subseteq h^{-1}[A \cap D^{\mathcal{B}}] = h^{-1}[D^{\mathcal{A}}]$.

Conversely, assume $(\Gamma \cup \mathrm{SI}_\Sigma(\mathcal{A})) \vdash \varphi$ is not satisfied in some $\mathcal{B} \in \mathsf{M}$, in which case there is some $h \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{B})$ such that $(\Gamma \cup \mathrm{SI}_\Sigma(\mathcal{A})) \subseteq h^{-1}[D^{\mathcal{B}}] \not\ni \varphi$, and so $\mathcal{A} \triangleq (\mathcal{B}{\upharpoonright}(\mathrm{img}\, h))$ is a submatrix of $\mathcal{B}$, $h \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{A})$ is surjective and $(\Gamma \cup \mathrm{SI}_\Sigma(\mathcal{A})) \subseteq h^{-1}[D^{\mathcal{B}}] = h^{-1}[A \cap D^{\mathcal{B}}] = h^{-1}[D^{\mathcal{A}}] \not\ni \varphi$. Finally, consider any $\psi \in \mathcal{A}$ and any $g \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{A})$. Then, as $(\mathrm{img}\, h) = A$, there is some Σ-substitution σ such that $g = (h \circ \sigma)$, in which case $g(\psi) = h(\sigma(\psi)) \in h[\mathrm{SI}_\Sigma(\mathcal{A})] \subseteq D^{\mathcal{A}}$, and so ψ is satisfied in $\mathcal{A}$. Thus, $\mathcal{A}$, being consistent, for $h(\varphi) \in (A \setminus D^{\mathcal{A}})$, belongs to S, as required. ☐

3. Sequent calculi and their semantics

3.1. **Two-side sequent calculi.** Here, we entirely follow the universal formalism of [9] but language/signature notation and with using rather $\rightarrowtail$ than $\vdash$ as the side separator symbol to avoid confusion with notations adopted in [12]. This, in particular, concerns the notions of *(propositional two-side)* Σ*-sequent rule/calculus* (/referred to as a *deductive base* therein), rule *derivable/admissible in* a calculus (/said to be *permissible* therein) as well as the *consequence* (viz., *derivability*) closure operator $\mathrm{Cn}_{\mathbb{C}}$ of (the *logical system defined by*) a calculus $\mathbb{C}$.

3.1.1. *Multiplicative calculi with structural rules versus two-side matrices.* First, *(basic) structural* rules are Σ-sequent rules of the form:

	Left	Right
Enlargement	$\dfrac{\Gamma \rightarrowtail \Delta}{\Gamma, \phi \rightarrowtail \Delta}$	$\dfrac{\Gamma \rightarrowtail \Delta}{\Gamma \rightarrowtail \Delta, \phi}$
Permutation	$\dfrac{\Gamma, \phi, \psi, \Theta \rightarrowtail \Delta}{\Gamma, \psi, \phi, \Theta \rightarrowtail \Delta}$	$\dfrac{\Gamma \rightarrowtail \Delta, \phi, \psi, \Theta}{\Gamma \rightarrowtail \Delta, \psi, \phi, \Theta}$
Contraction	$\dfrac{\Gamma, \phi, \phi \rightarrowtail \Delta}{\Gamma, \phi \rightarrowtail \Delta}$	$\dfrac{\Gamma \rightarrowtail \Delta, \phi, \phi}{\Gamma \rightarrowtail \Delta, \phi}$

where $\phi, \psi \in \mathrm{Fm}_\Sigma$ and $\Gamma, \Delta, \Theta \in \mathrm{Fm}^*_\Sigma$. Next, *Sharings* are axioms of the form $\varphi \rightarrowtail \varphi$, where $\varphi \in \mathrm{Fm}_\Sigma$. Likewise, *[weak /ortho-]Cuts* are rules of the form:

$$\frac{\Gamma \rightarrowtail \Xi, \varphi; \varphi, \Theta \rightarrowtail \Delta}{\Gamma, \Theta \rightarrowtail \Xi, \Delta}$$

where $\varphi \in \mathrm{Fm}_\Sigma$ and $\Gamma, \Delta, \Xi, \Theta \in \mathrm{Fm}^*_\Sigma$ [whereas each of them/either Ξ or Θ is empty]. By $\mathbb{S}_{(\mathrm{S})[\{\mathrm{W/O}\}\mathrm{C}]}$ we denote the two-side propositional Σ-sequent calculus constituted by basic structural rules (and Sharings) [as well as {weak /ortho-}Cuts].

Further, a propositional Σ-sequent calculus $\mathbb{C}$ is said to be *multiplicative*, provided, for any $\frac{Y}{\Gamma \rightarrowtail \Delta} \in \mathbb{C}$ and all $\Lambda, \Omega \in \mathrm{Fm}^*_\Sigma$, the rule

$$\frac{\{\Theta, \Lambda \rightarrowtail \Xi, \Omega \mid (\Theta \rightarrowtail \Xi) \in Y\}}{\Gamma, \Lambda \rightarrowtail \Delta, \Omega}$$

is derivable in $\mathbb{C}$.

By a *two-side Σ-matrix* we mean any triple of the form $\mathcal{A} \triangleq \langle \mathfrak{A}, L^{\mathcal{A}}, R^{\mathcal{A}} \rangle$ denoted by a Calligraphic letter (possibly, with indices), where $\mathfrak{A}$ is a Σ-algebra, called the *underlying* one of $\mathcal{A}$ and denoted by the corresponding Fraktur letter (with same indices, if any), and $L^{\mathcal{A}}, R^{\mathcal{A}} \subseteq A$, to be treated as a model structure of the first-order signature $\Sigma \cup \{L, R\}$ with unary *left-/right-truth* predicate L/R, any propositional Σ-sequent $\Gamma \rightarrowtail \Delta$, where $\Gamma, \Delta \in \mathrm{Fm}^*_\Sigma$, being identified with the first-order clause $\bigvee(\neg[L[\mathrm{img}\,\Gamma]] \cup R[\mathrm{img}\,\Delta])$, sequent rules being identified with respective implications of clauses.[1] Then, $\mathcal{A}$ is said to be *reflexive/transitive*, provided $L^{\mathcal{A}} \subseteq / \supseteq R^{\mathcal{A}}$, respectively, that is, Sharings/[weak] Cuts are true in $\mathcal{A}$. When $\mathcal{A}$ is both reflexive and transitive, we come to the standard matrix approach to sequents adopted in [10] and [16].

A class M of two-side Σ-matrices is said to be *(strongly) characteristic for* a propositional Σ-sequent calculus $\mathbb{C}$ iff any Σ-sequent rule is derivable in $\mathbb{C}$ iff it is true in M.[2] As a particular case of what was *actually* proved in Theorem 4.6 of [15], we then have:

Lemma 3.1. *A propositional two-side Σ-sequent calculus $\mathbb{C}$ has a characteristic class of (reflexive) [transitive] two-side Σ-matrices {with underlying algebra $\mathfrak{Fm}_\Sigma$} iff it is multiplicative and basic structural rules (and Sharings) [as well as Cuts] are derivable in $\mathbb{C}$.*

The "()[]"-optional particular case of Lemma 3.1 has been due to [13].

In view of Enlargement and Permutation, $\mathbb{S}_{(\mathrm{S})[\mathrm{C}]}$ is multiplicative. Therefore, as an immediate consequence of Lemma 3.1, we have:

Corollary 3.2. *The class of all (reflexive) [transitive] two-side Σ-matrices {with underlying algebra $\mathfrak{Fm}_\Sigma$} is characteristic for $\mathbb{S}_{(\mathrm{S})[\mathrm{C}]}$.*

It is this corollary that enables us to find semantics of non-multiplicative sequent calculi with structural rules below following [13].

3.1.2. *Fuzzy two-side matrices: beyond multiplicativity.* A *[distributive] fuzzy two-side Σ-matrix* is any tetrad of the form $\mathcal{A} = \langle \mathfrak{A}, \mathfrak{L}^{\mathcal{A}}, \lambda^{\mathcal{A}}, \mu^{\mathcal{A}} \rangle$ denoted by a Calligraphic letter (possibly, with indices), where $\mathfrak{A}$ is a Σ-algebra, called the *underlying* one of $\mathcal{A}$ and denoted by the corresponding Fraktur letter (with same indices, if any), $\mathfrak{L}^{\mathcal{A}}$ is a bounded [distributive] lattice, called the *truth* one of $\mathcal{A}$,

[1]In this way, here, we actually follow a semantic approach to sequents being *left-dual* to that of [15] in the sense of involving the complementary left-truth predicate instead of the falsehood one to make both the standard matrix approach adopted in [10] and [16] a particular case of that developed here and further "fuzzyfication" along the line of [13] more natural and immediate.

[2]This terminology is equally adopted *mutatis mutandis* within the context of fuzzy two-side matrices as well as that of both (fuzzy) signed matrices and signed sequent calculi.

and $(\lambda/\mu)^{\mathcal{A}} : A \to L^{\mathcal{A}}$, called the *left/right membership function of* $\mathcal{A}$. This is said to be *crisp* (viz., *bi-valent*), whenever $\mathfrak{L}^{\mathcal{A}} = \mathfrak{D}_{2,01}$, in which case it is identified with the ordinary two-side Σ-matrix with the same underlying algebra and the left/right truth predicate $((\lambda/\mu)^{\mathcal{A}})^{-1}[\{1\}]$. Likewise, it is said to be *reflexive/transitive*, provided $\lambda^{\mathcal{A}}(a)(\leqslant/\geqslant)^{\mathfrak{L}^{\mathcal{A}}}\mu^{\mathcal{A}}(a)$, for all $a \in A$. (Both reflexive and transitive [distributive] fuzzy two-side Σ-matrices are actually fuzzy Σ-matrices in the sense of [14] [resp., [13]].) Then, a Σ-sequent $\Gamma \rightarrowtail \Delta$, where $\Gamma, \Delta \in \mathrm{Fm}_{\Sigma}^{*}$, is said to be *true in* $\mathcal{A}$ *under* $h \in \hom(\mathfrak{Fm}_{\Sigma}, \mathfrak{A})$ ($\mathcal{A} \models (\Gamma \rightarrowtail \Delta)[h]$, in symbols), provided $(\bigwedge^{\mathfrak{L}^{\mathcal{A}}} \lambda^{\mathcal{A}}[h[\mathrm{img}\,\Gamma]]) \leqslant^{\mathfrak{L}^{\mathcal{A}}} (\bigvee^{\mathfrak{L}^{\mathcal{A}}} \mu^{\mathcal{A}}[h[\mathrm{img}\,\Delta]])$. (This fits well the crisp case as well as the both reflexive and transitive one. In particular, secondary model-theoretic notions are supposed to be clear without explicit specifying these.)

Fuzzy two-side Σ-matrices are nothing but *heterogeneous* algebras over a two-sort scheme (cf., e.g., [20]). In particular, given a set I and an I-tuple $\overline{\mathcal{A}}$ of [distributive] fuzzy two-side Σ-matrices, we have its *fuzzy direct product* $\bigotimes \overline{\mathcal{A}} = \bigotimes_{i\in I} \mathcal{A}_i$, being the [distributive] fuzzy two-side Σ-matrix with underlying algebra $\prod_{i\in I} \mathfrak{A}_i$, truth lattice $\prod_{i\in I} \mathfrak{L}_{\mathcal{A}_i}$ and membership functions defined point-wise: $(\lambda/\mu)^{\bigotimes \overline{\mathcal{A}}} : \bar{a} \mapsto \langle (\lambda/\mu)^{\mathcal{A}_i}(a_i)\rangle_{i\in I}$. (As usual, in case $I = 2$, $\mathcal{A}_0 \otimes \mathcal{A}_1$ stands for the fuzzy direct product involved.)

Theorem 3.3. *A propositional two-side* Σ*-sequent calculus* $\mathbb{C}$ *has a characteristic class of (reflexive) [transitive distributive] fuzzy two-side* Σ*-matrices {with underlying algebra* $\mathfrak{Fm}_{\Sigma}$*} iff basic structural rules (and Sharings) [as well as Cuts] are derivable in* $\mathbb{C}$.

Proof. The "only if" part is immediate. Conversely, assume basic structural rules (and Sharings) [as well as Cuts] are derivable in $\mathbb{C}$. Then, $(\mathrm{img}\,\mathrm{Cn}_{\mathbb{C}}) \subseteq \mathcal{C} \triangleq (\mathrm{img}\,\mathrm{Cn}_{\mathbb{S}_{(\mathrm{S})[\mathrm{C}]}})$. Moreover, by Corollary 3.2, there is a *set* — viz., not a proper class — M of (reflexive) [transitive] two-side Σ-matrices with underlying algebra $\mathfrak{Fm}_{\Sigma}$ characteristic for $\mathbb{S}_{(\mathrm{S})[\mathrm{C}]}$. Given a fuzzy two-side Σ-matrix $\mathcal{A}$ with underlying algebra $\mathfrak{Fm}_{\Sigma}$, by $S^{\mathcal{A}}$ we denote the set of all propositional two-side Σ-sequents true in $\mathcal{A}$ under the diagonal Σ-substitution. Then, taking the fact that the set of all rules derivable in a calculus is closed under substitutions into account, $\{S^{\mathcal{A}} \mid \mathcal{A} \in \mathsf{M}\}$ is a basis of $\mathcal{C}$. Consider any $X \in (\mathrm{img}\,\mathrm{Cn}_{\mathbb{C}}) \subseteq \mathcal{C}$. Then, by the Choice Axiom, there is some $I \subseteq \mathsf{M}$ such that $X = (\mathrm{Seq}_{\Sigma} \cap \bigcap_{i\in I} S^{i})$. Let $\mathcal{A}_X$ be the (reflexive) [transitive] distributive fuzzy two-side Σ-matrix with underlying algebra $\mathfrak{Fm}_{\Sigma}$, truth lattice $\mathfrak{D}_{2,01}^{I}$ and left/right membership function $\prod_{i\in I} \chi_{\mathrm{Fm}_{\Sigma}}^{(L/R)^{i}}$. Clearly, $X = S^{\mathcal{A}_X}$. In this way, taking the fact that the set of all rules derivable in a calculus is closed under substitutions into account, we eventually conclude that $\{\mathcal{A}_X \mid X \in (\mathrm{img}\,\mathrm{Cn}_{\mathbb{C}})\}$ is characteristic for $\mathbb{C}$, as required. □

The "()[]"-optional particular case of Theorem 3.3 has been due to [13]. In this connection, it is remarkable that the argumentation of [13] is essentially based upon presence of Sharings and (at least, ortho-)Cuts, so it is not applicable to proving Theorem 3.3 in general. In this way, since the "multiplicative-crisp" case was derived therein from the fuzzy one and, what is more, with using Reflexivity, Lemma 3.1 equally occurs to be essentially beyond the scopes of [13]. This highlights the substantial advance of the present study with regard to [13].

3.2. **Signed sequent calculi.** Here, we entirely follow the formalism of [15] but denote signatures/languages/variables like in [12] and, for covering non-multiplicative calculi (equivalent to the *many-place sequent* ones in the sense of [19]), deal with signed sequent substitutions with merely empty right components, becoming thus essentially identical to their left components being ordinary Σ-substitutions.

Under this alteration, *(basic) structural* rules are those of the form (cf. Definition 3.14 of [15])):

$$\frac{\Gamma}{\Gamma \cup \{s : x\}}$$

where $s \in \mathsf{S}$ and $\Gamma \in \wp_\omega(\mathsf{S} : \mathrm{Fm}_\Sigma)$. Likewise (cf. Definition 3.15 of [15]), given any $\mathcal{S} \subseteq \wp(\mathsf{S})$, $\mathcal{S}$-*Sharings/-Cuts* are axioms/rules of the form:

$$(\Gamma \cup (S : \{x\}) / \frac{\{\Gamma \cup \{s : x\} \mid s \in S\}}{\Gamma}$$

where $S \in \mathcal{S}$ and $\Gamma \in \wp_\omega(\mathsf{S} : \mathrm{Fm}_\Sigma)$. Finally, an S-signed Σ-sequent calculus $\mathbb{C}$ is said to be *multiplicative*, provided, for every $\frac{X}{\Delta} \in \mathbb{C}$ and all $\Gamma \in \wp_\omega(\mathsf{S} : \mathrm{Fm}_\Sigma)$, the rule $\frac{\{\Xi \cup \Gamma \mid \Xi \in X\}}{\Delta \cup \Gamma}$ is derivable in $\mathbb{C}$.

Given any $\mathcal{S} \subseteq \wp(\mathsf{S})$, an S-signed Σ-matrix $\mathcal{A}$ is said to be $\mathcal{S}$-*reflexive/-transitive*, provided, for all $S \in \mathcal{S}$, $(A \cap ((\bigcup / \bigcap)\nabla[S])) = (A/\varnothing)$, that is, $\mathcal{S}$-Sharings/-Cuts are true in $\mathcal{A}$.

Under the above conventions, the factual content of Theorem 4.6 of [15] is formulated as follows:

Lemma 3.4. *Let $\mathcal{S}, \mathcal{T} \subseteq \wp(\mathsf{S})$. Then, a propositional S-signed Σ-sequent calculus $\mathbb{C}$ has a characteristic class of $\mathcal{S}$-reflexive $\mathcal{T}$-transitive S-signed Σ-matrices (with underlying algebra $\mathfrak{Fm}_\Sigma$) iff it is multiplicative and basic structural rules, $\mathcal{S}$-Sharings and $\mathcal{T}$-Cuts are derivable in $\mathbb{C}$.*

Given any $\mathcal{S}, \mathcal{T} \subseteq \wp(\mathsf{S})$, by $\mathbb{S}_{\mathcal{S},\mathcal{T}}$ we denote the signed sequent calculus constituted by all basic structural rules, $\mathcal{S}$-Sharings and $\mathcal{T}$-Cuts. Clearly, it is multiplicative. Therefore, as an immediate consequence of Lemma 3.4, we have:

Corollary 3.5. *Let $\mathcal{S}, \mathcal{T} \subseteq \wp(\mathsf{S})$. Then, the class of all $\mathcal{S}$-reflexive $\mathcal{T}$-transitive S-signed Σ-matrices (with underlying algebra $\mathfrak{Fm}_\Sigma$) is characteristic for $\mathbb{S}_{\mathcal{S},\mathcal{T}}$.*

3.2.1. *Fuzzy signed matrices: beyond multiplicativity.* A *[distributive] fuzzy S-signed Σ-matrix* is any triple of the form $\mathcal{A} = \langle \mathfrak{A}, \mathfrak{L}^{\mathcal{A}}, \mu^{\mathcal{A}} \rangle$ denoted by a Calligraphic letter (possibly, with indices), where $\mathfrak{A}$ is a Σ-algebra, called the *underlying* one of $\mathcal{A}$ and denoted by the corresponding Fraktur letter (with same indices, if any), $\mathfrak{L}^{\mathcal{A}}$ is a [distributive] bounded lattice, called the *truth* one of $\mathcal{A}$, and $\mu^{\mathcal{A}} : (\mathsf{S} \times A) \to L^{\mathcal{A}}$, called the *($\mathsf{S}$-signed) membership function of* $\mathcal{A}$. This is said to be *crisp* (viz., *bi-valent*), whenever $\mathfrak{L}^{\mathcal{A}} = \mathfrak{D}_{2,01}$, in which case it is identified with the ordinary S-signed Σ-matrix $\langle \mathfrak{A}, \langle \{a \in A \mid \mu^{\mathcal{A}}(s,a) = 1\} \rangle_{s \in \mathsf{S}} \rangle$. Likewise, given any $\mathcal{S} \subseteq \wp(\mathsf{S})$, $\mathcal{A}$ is said to be $\mathcal{S}$-*reflexive/-transitive*, provided, for all $a \in A$, it holds that $((\bigvee / \bigwedge)^{\mathfrak{L}^{\mathcal{A}}}\{\mu^{\mathcal{A}}(s,a) \mid s \in \mathcal{S}\}) = (\top/\bot)^{\mathfrak{L}^{\mathcal{A}}}$. Then, an S-signed Σ-sequent Γ is said to be *true in $\mathcal{A}$ under* $h \in \hom(\mathfrak{Fm}_\Sigma, \mathfrak{A})$, provided $(\bigvee^{\mathfrak{L}^{\mathcal{A}}} \mu^{\mathcal{A}}[\{\langle s, h(\varphi)\rangle \mid (s : \varphi) \in \Gamma\}]) = \top^{\mathfrak{L}^{\mathcal{A}}}$. (This fits well the crisp case of [15]. In particular, secondary model-theoretic notions are supposed to be clear without explicit specifying these.)

Theorem 3.6. *Let $\mathcal{S}[, \mathcal{T}] \subseteq \wp(\mathsf{S})$. Then, a propositional S-signed Σ-sequent calculus $\mathbb{C}$ has a characteristic class of $\mathcal{S}$-reflexive [$\mathcal{T}$-transitive distributive] fuzzy S-signed Σ-matrices {with underlying algebra $\mathfrak{Fm}_\Sigma$} iff basic structural rules, $\mathcal{S}$-Sharings [and $\mathcal{T}$-Cuts] are derivable in $\mathbb{C}$.*

Proof. The "only if" part is immediate. Conversely, assume basic structural rules, $\mathcal{S}$-Sharings [and $\mathcal{T}$-Cuts] are derivable in $\mathbb{C}$. Then, we have $(\mathrm{img}\,\mathrm{Cn}_{\mathbb{C}}) \subseteq \mathcal{C} \triangleq (\mathrm{img}\,\mathrm{Cn}_{\mathbb{S}_{\mathcal{S},\varnothing[\cup\mathcal{T}]}})$. Moreover, by Corollary 3.2, there is a *set* — viz., not a proper class — M of $\mathcal{S}$-reflexive [$\mathcal{T}$-transitive] S-signed Σ-matrices with underlying algebra $\mathfrak{Fm}_\Sigma$ characteristic for $\mathbb{S}_{\mathsf{S},\varnothing[\mathrm{cup}\mathcal{T}]}$. Given a fuzzy S-signed Σ-matrix $\mathcal{A}$ with underlying algebra $\mathfrak{Fm}_\Sigma$, by $S^{\mathcal{A}}$ we denote the set of all S-signed Σ-sequents true in $\mathcal{A}$ under the diagonal Σ-substitution. Then, taking the fact that the set of all rules derivable in a calculus is closed under substitutions into account, $\{S^{\mathcal{A}} \mid \mathcal{A} \in \mathsf{M}\}$ is a basis of $\mathcal{C}$. Consider any $X \in (\mathrm{img}\,\mathrm{Cn}_{\mathbb{C}}) \subseteq \mathcal{C}$. Then, by the Choice Axiom, there is some $I \subseteq \mathsf{M}$ such that $X = (\mathrm{Seq}_\Sigma \cap \bigcap_{i \in I} S^i)$. Let $\mathcal{A}_X$ be the $\mathcal{S}$-reflexive [$\mathcal{T}$-transitive] distributive fuzzy S-signed Σ-matrix with underlying algebra $\mathfrak{Fm}_\Sigma$, truth lattice $\mathfrak{D}^I_{2,01}$ and membership function $\prod_{i \in I} \chi^{\bigcup_{s \in \mathsf{S}}(\{s\} \times \nabla^i(s))}_{\mathsf{S}:\,\mathrm{Fm}_\Sigma}$. Clearly, $X = S^{\mathcal{A}_X}$. In this way, taking the fact that the set of all rules derivable in a calculus is closed under substitutions into account, we eventually conclude that $\{\mathcal{A}_X \mid X \in (\mathrm{img}\,\mathrm{Cn}_{\mathbb{C}})\}$ is characteristic for $\mathbb{C}$, as required. □

The methodological value of Theorems 3.3 and 3.6 is that their proofs make it clear, in general, how fuzzification of crisp model structures providing semantics of solely multiplicative calculi with structural rules (in particular, those constituted by merely structural rules) yields semantics of non-multiplicative calculi with structural rules. This point going back to [17] is essentially beyond the scopes of [13] and resembles the underlying idea of [12] extended to *heterogeneous* algebras, because, like fuzzy two-side matrices, fuzzy signed ones are equally such algebras in their substance.

4. Applications to substructural versions of *LK*

From now on, we deal with the signature $\Sigma_{[01]} \triangleq (\Sigma_{+[,01]} \cup \{\neg\})$, where $\neg$ (negation) is unary, entirely following Subsection 3.1.

[Bounded] De Morgan/Kleene/ Boolean lattices (cf. [11] and [10])[3] are supposed to be of the signature $\Sigma_{[01]}$ [the variety of all them being denoted by $\mathsf{DMA}/\mathsf{KA}/\mathsf{BA}$, respectively]. Given any $n \in (\omega \setminus 2)$, by $\mathfrak{K}_{n[,01]}$ we denote the [bounded] Kleene lattice such that $(\mathfrak{K}_{n[,01]} {\restriction} \Sigma_{+[,01]}) \triangleq \mathfrak{D}_{n[,01]}$ and $\neg^{\mathfrak{K}_{n[,01]}} i \triangleq (n-1-i)$, for all $i \in n$, in which case $e_n : 2 \to n, j \mapsto ((n-1) \cdot j)$ is an embedding of $\mathfrak{B}_2 \triangleq \mathfrak{K}_{2,01} \in \mathsf{BA}$ into $\mathfrak{K}_{n,01}$.

By $\mathfrak{DM}_{4[,01]}$ we denote the [bounded] De Morgan lattice such that $(\mathfrak{DM}_{4[,01]} {\restriction} \Sigma_{+[,01]}) \triangleq \mathfrak{D}^2_{2[,01]}$ and $\neg^{\mathfrak{DM}_{4[,01]}} \langle i, j \rangle \triangleq \langle 1-j, 1-i \rangle$, for all $i, j \in 2$, in which case, for every $k \in 2$, $e_{3,k} : 3 \to 2^2, l \mapsto \{\langle 1-k, l-[l/2] \rangle, \langle k, [l/2] \rangle\}$ is an embedding of $\mathfrak{K}_{3[,01]}$ into $\mathfrak{DM}_{4[,01]}$, and so, for any $\vec{m} \in 2^*$, we have the subalgebra $\mathfrak{DM}_{4[,01]-\vec{m}} \triangleq (\mathfrak{DM}_{4[,01]} {\restriction} (2^2 \cap \bigcap_{k \in \mathrm{img}\,\vec{m}} (\mathrm{img}\, e_{3,k})))$ of $\mathfrak{DM}_{4[,01]}$, $\mathfrak{DM}_{4[,01]-k}$ being isomorphic to $\mathfrak{K}_{3[,01]}$ under $e_{3,k}$, $\mathfrak{DM}_{4[,01]-01}$ being isomorphic to $\mathfrak{K}_{2[,01]}$ under $e_{3,k} \circ e_3$, where $k \in 2$. In this connection, we use the following standard abbreviations going back to [2]:

$$\begin{aligned} \mathsf{t} &\triangleq \langle 1, 1 \rangle, \\ \mathsf{f} &\triangleq \langle 0, 0 \rangle, \\ \mathsf{b} &\triangleq \langle 1, 0 \rangle, \\ \mathsf{n} &\triangleq \langle 0, 1 \rangle. \end{aligned}$$

Then, for any $\vec{c} \in \{\mathsf{n}, \mathsf{b}\}^*$, we set $\mathfrak{DM}_{4[,01]-\vec{c}} \triangleq \mathfrak{DM}_{4[,01]-(\pi_1 \circ \vec{c})}$, in which case $DM_{4[,01]-\vec{c}} = (2^2 \setminus (\mathrm{img}\,\vec{c}))$.

By $\mathbb{LK}_{(\mathrm{S})[\{\mathrm{W/O}\}\mathrm{C}]}$ we denote the two-side propositional Σ_{01}-sequent calculus constituted by basic structural rules (and Sharings) [as well as {weak /ortho-}Cuts]

[3]Bounded De Morgan/Kleene/Boolean lattices are traditionally called *De Morgan/Kleene/Boolean algebras* (cf. [1]).

and the following rules collectively with inverse to these:

$$
\begin{array}{ccc}
 & \text{Left} & \text{Right} \\
(\wedge) & \dfrac{\Gamma,\phi,\psi \rightarrowtail \Delta}{\Gamma,\phi\wedge\psi \rightarrowtail \Delta} & \dfrac{\Gamma \rightarrowtail \Delta,\phi;\Gamma \rightarrowtail \Delta,\psi}{\Gamma \rightarrowtail \Delta,\phi\wedge\psi} \\
(\vee) & \dfrac{\Gamma,\phi \rightarrowtail \Delta;\Gamma,\psi \rightarrowtail \Delta}{\Gamma,\phi\vee\psi \rightarrowtail \Delta} & \dfrac{\Gamma \rightarrowtail \Delta,\phi,\psi}{\Gamma \rightarrowtail \Delta,\phi\vee\psi} \\
(\neg) & \dfrac{\Gamma \rightarrowtail \Delta,\phi}{\Gamma,\neg\phi \rightarrowtail \Delta} & \dfrac{\Gamma,\phi \rightarrowtail \Delta}{\Gamma \rightarrowtail \Delta,\neg\phi} \\
(\bot) & \Gamma,\bot \rightarrowtail \Delta & \dfrac{\Gamma \rightarrowtail \Delta,\bot}{\Gamma \rightarrowtail \Delta} \\
(\top) & \dfrac{\Gamma,\top \rightarrowtail \Delta}{\Gamma \rightarrowtail \Delta} & \Gamma \rightarrowtail,\Delta,\top
\end{array}
$$

where $\phi,\psi \in \mathrm{Fm}_{\Sigma_{01}}$ and $\Gamma,\Delta \in \mathrm{Fm}^*_{\Sigma_{01}}$. Then, $\mathbb{LK}_{\mathrm{SC}}$ is the propositional fragment of Gentzen' calculus [3] supplemented with rules inverse to the above *logical* ones that are derivable in the original calculus, so they have same derivable rules, though such is the case for the neither Cut- nor Sharing-free versions.

Remark 4.1. A two-side Σ_{01}-matrix $\mathcal{A}$ is a model of $\mathbb{LK}$ iff the following hold:

$$(\neg^{\mathfrak{A}} a \in (R/L)^{\mathcal{A}}) \;\Leftrightarrow\; (a \notin (L/R)^{\mathcal{A}}), \tag{4.1}$$

$$\bot^{\mathfrak{A}} \;\notin\; X, \tag{4.2}$$

$$\top^{\mathfrak{A}} \;\in\; X, \tag{4.3}$$

$$((a \wedge^{\mathfrak{A}} b) \in X) \;\Leftrightarrow\; (\{a,b\} \subseteq X), \tag{4.4}$$

$$((a \vee^{\mathfrak{A}} b) \in X) \;\Leftrightarrow\; ((\{a,b\} \cap X) \neq \varnothing), \tag{4.5}$$

for all $a,b \in A$ and all $X \in \{L^{\mathcal{A}}, R^{\mathcal{A}}\}$. □

Put $\mathcal{DM}_4 \triangleq \langle \mathfrak{DM}_{4,01}, \{\mathsf{t},\mathsf{n}\}, \{\mathsf{t},\mathsf{b}\}\rangle$ and $\mathcal{DM}_{4-\vec{b}} \triangleq (\mathcal{DM}_4 \restriction DM_{4-\vec{b}})$, where $\vec{b} \in (2^* \cup \{\mathsf{n},b\}^*)$.

Theorem 4.2 (Strong Completeness Theorem)**.** *A Σ_{01}-sequent rule is derivable in $\mathbb{LK}_{(\mathrm{S})[\mathrm{C}]}$ iff it is true in $\mathcal{DM}_{4-(\mathsf{n})[\mathsf{b}]}$.*

Proof. We use Remark 4.1 tacitly. Then, $\mathcal{DM}_{4-(\mathsf{n})[\mathsf{b}]}$, being (reflexive) [transitive], is clearly a model of $\mathbb{LK}_{(\mathrm{S})[\mathrm{C}]}$. Conversely, $\mathbb{LK}_{(\mathrm{S})[\mathrm{C}]}$ is multiplicative, in view of Enlargement and Permutation, and contains all basic structural rules (and Sharings) [as well as Cuts], in which case, by Lemma 3.1, it has a characteristic class M of (reflexive) [transitive] two-side Σ_{01}-matrices. Consider any $\mathcal{A} \in \mathsf{M}$. Then, by (4.2), (4.3), (4.4) and (4.5), we see that, for every $X \in \{L^{\mathcal{A}}, R^{\mathcal{A}}\}$, $\chi^X_A \in \hom(\mathfrak{A}\restriction\Sigma_{+,01}, \mathfrak{D}_{2,01})$, in which case $h \triangleq (\chi^{R^{\mathcal{A}}}_A \times \chi^{L^{\mathcal{A}}}_A) \in \hom(\mathfrak{A}\restriction\Sigma_{+,01}, \mathfrak{D}^{2,01}_2)$, and so, by (4.1), we eventually conclude that $h \in \hom(\mathfrak{A}, \mathfrak{DM}_{4,01})$. And what is more, $L^{\mathcal{A}} = h^{-1}[\{\mathsf{t},\mathsf{n}\}]$ and $R^{\mathcal{A}} = h^{-1}[\{\mathsf{t},\mathsf{b}\}]$. Finally, once $\mathcal{A}$ is (reflexive) [transitive], we also have $(\mathsf{n})[\mathsf{b}] \notin (\mathrm{img}\, h)$. Thus, h is a homomorphism from $\mathcal{A}$ to $\mathcal{DM}_{4-(\mathsf{n})[\mathsf{b}]}$. Hence, by Proposition 4.3 of [15], any Σ_{01}-sequent rule, being true in $\mathcal{DM}_{4-(\mathsf{n})[\mathsf{b}]}$, is true in $\mathcal{A}$, as required. □

Since $\mathfrak{B}_2$ is isomorphic to $\mathfrak{DM}_{4,01-\mathsf{nb}}$, in view of Proposition 4.4 of [15], the double-optional case of Theorem 4.2 incorporates the well-known strong completeness theorem for LK having same derivable rules as $\mathbb{LK}_{\mathrm{SC}}$ (cf., e.g., [16]).

Let

$$\tau : \mathrm{Seq}_{\Sigma_{01}} \to \mathrm{Fm}_{\Sigma_{01}}, (\Gamma \rightarrowtail \Delta) \to \begin{cases} \bot & \text{if } \Gamma = \Delta = \varnothing, \\ \vee((\neg \circ \Gamma), \Delta) & \text{otherwise.} \end{cases}$$

and $\rho : \mathrm{Fm}_{\Sigma 01} \to \mathrm{Seq}_{\Sigma_{01}}, \varphi \to (\varnothing \rightarrowtail \varphi)$ (these are actually *translations* in the sense of [9]). Then, the following key auxiliary observation is immediate:

Lemma 4.3. *Let* $\Gamma, \Delta \in \mathrm{Fm}^*_{\Sigma_{01}}$, $\varphi \in \mathrm{Fm}_{\Sigma_{01}}$ *and* $h \in \hom(\mathfrak{Fm}_{\Sigma_{01}}, \mathfrak{DM}_{4,01})$. *Then,* $(h(\varphi) \in \{\mathsf{t}, \mathsf{b}\}) \Leftrightarrow (\mathcal{DM}_4 \models \rho(\varphi)[h])$ *and* $(\mathcal{DM}_4 \models (\Gamma \rightarrowtail \Delta)[h]) \Leftrightarrow (h(\tau(\Gamma \rightarrowtail \Delta)) \in \{\mathsf{t}, \mathsf{b}\})$.

As a first immediate consequence of Lemma 4.3, we have:

Corollary 4.4. *Let* I *be a set,* $\overline{\mathcal{A}}$ *an* I*-tuple of submatrices of* $\mathcal{DM}_4$, $\Gamma, \Delta \in \mathrm{Fm}^*_{\Sigma_{01}}$, $\varphi \in \mathrm{Fm}_{\Sigma_{01}}$ *and* $h \in \hom(\mathfrak{Fm}_{\Sigma_{01}}, \prod_{i \in I} \mathfrak{A}_i)$. *Then,* $(h(\varphi) \in \{\mathsf{t}, \mathsf{b}\}^I) \Leftrightarrow ((\bigotimes \overline{\mathcal{A}}) \models \rho(\varphi)[h])$ *and* $((\bigotimes \overline{\mathcal{A}}) \models (\Gamma \rightarrowtail \Delta)[h]) \Leftrightarrow (h(\tau(\Gamma \rightarrowtail \Delta)) \in \{\mathsf{t}, \mathsf{b}\}^I)$.

Since $\mathfrak{K}_{3[,01]}|\mathfrak{K}_{2[,01]}$ is isomorphic to $\mathfrak{DM}_{4[,01]-(\mathsf{n}/\mathsf{b})|\mathsf{nb}}$, the logic of the logical $\Sigma_{[01]}$-matrix $\langle \mathfrak{DM}_{4[,01]-(\mathsf{n})\{\mathsf{b}\}}, \{\mathsf{t}, \mathsf{b}\}\{\setminus\{\mathsf{b}\}\}\rangle$ is nothing but [the bounded version of] Belnap's four-valued logic $B_{4[,01]}$ (cf. [2] and [7]) (resp., the *logic of paradox* $LP_{[01]}$ equally being the logic of $\langle \mathfrak{K}_{3[,01]}, 3 \setminus 1\rangle$; cf. [6] and [8]) {resp., Kleene's three-valued logic $K_{3[,01]}$ equally being the logic of $\langle \mathfrak{K}_{3[,01]}, 3 \setminus 2\rangle$; cf. [4]} ({resp., the classical logic $PC_{[01]}$ equally being the logic of $\langle \mathfrak{K}_{2[,01]}, 2 \setminus 1\rangle$}). In this way, by Theorem 4.2 and Lemma 4.3, we immediately get:

Corollary 4.5. $\mathbb{LK}_{(\mathrm{S})[\mathrm{C}]}$ *is equivalent in the sense of* [9] *with respect to* τ *and* ρ *to* $B_{4,01}$ *(*LP_{01}*) [*$K_{3,01}$*] ([*PC_{01}*]), respectively.*

The double-optional case of Corollary 4.5 yields a new insight into the equivalence of LK and PC (cf. [9]). Likewise, when restricting our consideration by merely non-empty sequents and Σ-formulas, the "()"-optional case of Corollary 4.5 collectively with Cut Elimination in $\mathbb{LK}_{\mathrm{SC}}$ (cf. Corollary 4.16) yield a new insight into the main result of [18]. In general, Corollary 4.5 collectively with [9] reduce the task of finding (axiomatic) extensions of $\mathbb{LK}$ to that of $B_{4,01}$. This clarifies the meaning of the next subsections and, in general, makes the problem of finding (axiomatic) extensions of $B_{4,01}$ especially acute.

4.1. **Prevarieties of Kleene algebras versus extensions of the logic of paradox and Kleene's three-valued logic.**

Here, we tacitly follow [12].[4] By K_n, where $n \in (\omega \setminus 2)$, we denote the prevariety generated by $\mathfrak{K}_{n,01}$. Clearly, $\mathsf{K}_n \subseteq \mathsf{KA}$ (although, in case $n = 3$, the converse inclusion is well known to hold as well, it is no matter for our further argumentation and, for this reason, is disregarded).

4.1.1. *Extensions of Kleene's three-valued logic.* Here, we deal with $\nabla \triangleq \{x \approx \top\}$. As $\mathfrak{K}^{\nabla}_{3,01} = \langle \mathfrak{K}_{3,01}, 3 \setminus 2\rangle$, to study extensions of $K_{3,01}$ is to study subprevarieties of K_3. We start from recalling the following well-known auxiliary observations:

Lemma 4.6. $\mathfrak{B}_2$ *is embeddable into any non-one-element De Morgan algebra* $\mathfrak{A}$.

Proof. In that case, $\bot^{\mathfrak{A}} \neq \top^{\mathfrak{A}}$, and so $\{\langle 0, \bot^{\mathfrak{A}}\rangle, \langle 1, \top^{\mathfrak{A}}\rangle\}$ is an embedding of $\mathfrak{B}_2$ into $\mathfrak{A}$, as required. □

Lemma 4.7. $\mathsf{BA} = \mathsf{K}_2$.

Proof. Consider any Boolean algebra $\mathfrak{A}$ and any distinct $a, b \in A$. Then, $c \triangleq (a \vee^{\mathfrak{A}} b) \nleq^{\mathfrak{A}} d \triangleq (a \wedge^{\mathfrak{A}} b)$. Therefore, by the Prime Ideal Theorem for distributive lattices (cf., e.g., [1]), there is some prime filter F of $\mathfrak{A} {\restriction} \Sigma_+$ such that $d \notin F \ni c$, in which case $(a \in F) \Leftrightarrow (b \notin F)$, while $\bot^{\mathfrak{A}} \notin F \ni \top^{\mathfrak{A}}$, and so $(e \in F) \Leftrightarrow (\neg^{\mathfrak{A}} e \notin F)$, for all $e \in A$. Hence, $h \triangleq \chi^F_A \in \hom(\mathfrak{A}, \mathfrak{B}_2)$. And what is more, $(h(a) = 1) \Leftrightarrow (h(b) = 0)$, in which case $h(a) \neq h(b)$, and so $\mathfrak{A} \in \mathsf{K}_2$, as required. □

[4]In this connection, we take the opportunity to notice that the term "prevariety" used therein, being a part of algebraic folklore within the former USSR, is due to [20]. Prevarieties are exactly implicational/abstract hereditary multiplicative classes in the sense of [11]/[5].

Theorem 4.8. *PC_{01} is the only consistent proper extension of $K_{3,01}$ and is relatively axiomatized by the Excluded Middle Law axiom $x \vee \neg x$. In particular, PC_{01} has no proper consistent extension.*

Proof. First, $\mathfrak{K}^{\nabla}_{3,01}\upharpoonright\{0,2\}$, being isomorphic to $\mathfrak{B}^{\nabla}_{2}$, defining PC_{01}, under e_3, is the only submatrix of $\mathfrak{K}^{\nabla}_{3,01}$ satisfying the axiom $x \vee \neg x$, for this is not satisfied in $\mathfrak{K}^{\nabla}_{3,01}$ under $[x/1]$. Hence, by Proposition 2.1, we conclude that PC_{01} is the proper axiomatic extension of $K_{3,01}$ relatively axiomatized by the axiom $x \vee \neg x$. Finally, consider any non-trivial prevariety $\mathsf{P} \subseteq \mathsf{K}_3$ and the following two complementary cases:

(1) $\mathsf{P} \subseteq \mathsf{BA}$.
Then, by Lemma 4.6, $\mathfrak{B}_2 \in \mathsf{P}$. Hence, by Lemma 4.7, the logic of P^{∇} is equal to PC_{01}.

(2) $\mathsf{P} \nsubseteq \mathsf{BA}$.
Take any $\mathfrak{A} \in (\mathsf{P} \setminus \mathsf{BA}) \neq \varnothing$. Then, there is some $a \in A$ such that $\neg^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} a \neq \top^{\mathfrak{A}}$. Consider the following complementary subcases:

(a) $\neg^{\mathfrak{A}} a = a$.
Then, the mapping $e : 3 \to A$ defined by:

$$\begin{aligned} e(0) &\triangleq \bot^{\mathfrak{A}}, \\ e(1) &\triangleq a, \\ e(2) &\triangleq \top^{\mathfrak{A}}, \end{aligned}$$

is an embedding of $\mathfrak{K}_{3,01}$ into $\mathfrak{A}$. Therefore, $\mathsf{P} = \mathsf{K}_3$. Hence, the logic of P^{∇} is equal to $K_{3,01}$.

(b) $\neg^{\mathfrak{A}} a \neq a$.
Then, the mapping $e : 4 \to A$ defined by:

$$\begin{aligned} e(0) &\triangleq \bot^{\mathfrak{A}}, \\ e(1) &\triangleq \neg^{\mathfrak{A}} a, \\ e(2) &\triangleq a, \\ e(3) &\triangleq \top^{\mathfrak{A}}, \end{aligned}$$

is an embedding of $\mathfrak{K}_{4,01}$ into $\mathfrak{A}$. Therefore, $\mathsf{P} \ni \mathfrak{K}_{4,01}$. Hence, the logic of $\mathfrak{K}^{\nabla}_{4,01}$ is an extension of the logic of P^{∇}, being, in its turn, an extension of $K_{3,01}$. On the other hand, $h : 4 \to 3, i \mapsto [(i+1)/2]$ is a surjective strict homomorphism from $\mathfrak{K}^{\nabla}_{4,01}$ onto $\mathfrak{K}^{\nabla}_{3,01}$, in which case their logics are equal, and so the logic of P^{∇} is equal to $K_{3,01}$.

Thus, in any subcase, the logic of P^{∇} is equal to $K_{3,01}$.

This completes the argument. □

Combining Corollary 4.5 and Theorem 4.8 with [9], we immediately get:

Corollary 4.9. *$\mathbb{LK}_{\mathrm{SC}}$ is the only proper consistent extension of $\mathbb{LK}_{\mathrm{C}}$. In particular, $\mathbb{LK}_{\mathrm{SC}}$ has no proper consistent extension.*

4.1.2. *Extensions of the logic of paradox.* Here, we deal with $\nabla \triangleq \{x \approx (x \vee \neg x)\}$. As $\mathfrak{K}^{\nabla}_{3,01} = \langle \mathfrak{K}_{3,01}, 3 \setminus 1 \rangle$, to study extensions of LP_{01} is to study subprevarieties of K_3.

By NP_{01} we denote the extension of LP_{01} relatively axiomatized by the *Ex Contradictione Quodlibet* rule:

$$\{x, \neg x\} \vdash y. \tag{4.6}$$

A Kleene algebra is said to be *non-paraconsistent*, provided the quasi-identity $\nabla(4.6)$ is true in it. The prevariety of all non-paraconsistent members of K_3 is denoted by NPK_3.

Lemma 4.10. *Let $\mathfrak{A}$ be a non-one-element non-paraconsistent Kleene algebra. Then,* $\hom(\mathfrak{A},\mathfrak{B}_2)\neq\varnothing$.

Proof. In that case, $F\triangleq\{a\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}a\mid a\in A\}\ni\top^{\mathfrak{A}}$ and $I\triangleq\{a\wedge^{\mathfrak{A}}\neg^{\mathfrak{A}}a\mid a\in A\}\ni\bot^{\mathfrak{A}}$ are disjoint filter and ideal, respectively, of the distributive lattice $\mathfrak{A}{\restriction}\Sigma_+$. Therefore, by the Prime Ideal Theorem for distributive lattices (cf., e.g., [1]), there is a prime filter $G\supseteq F$ of $\mathfrak{A}{\restriction}\Sigma_+$ disjoint with I, in which case $(a\in G)\Leftrightarrow(\neg^{\mathfrak{A}}a\notin G)$, for all $a\in A$, and so $\chi^G_A\in\hom(\mathfrak{A},\mathfrak{B}_2)$, as required. □

Proposition 4.11. NPK_3 *is generated by* $\mathfrak{K}_{3,01}\times\mathfrak{B}_2$.

Proof. Clearly, $(\mathfrak{K}_{3,01}\times\mathfrak{B}_2)\in\mathsf{NPK}_3$. Conversely, consider any $\mathfrak{A}\in\mathsf{NPK}_3$ and any distinct $a,b\in A$, in which case $\mathfrak{A}$ is not one-element, and so, by Lemma 4.10, there is some $g\in\hom(\mathfrak{A},\mathfrak{B}_2)$. Then, there is also some $h\in\hom(\mathfrak{A},\mathfrak{K}_{3,01})$ such that $h(a)\neq h(b)$, in which case $f\triangleq(h\times g)\in\hom(\mathfrak{A},\mathfrak{K}_{3,01}\times\mathfrak{B}_2)$ and $f(a)\neq f(b)$, and so $\mathfrak{A}$ belongs to the prevariety generated by $\mathfrak{K}_{3,01}\times\mathfrak{B}_2$, as required. □

As a consequence of Proposition 4.11, we have:

Corollary 4.12. NP_{01} *is defined by* $\langle\mathfrak{K}_{3,01},3\setminus 1\rangle\times\langle\mathfrak{B}_2,2\setminus 1\rangle$.

By MP_{01} we denote the extension of LP_{01} relatively axiomatized by the *Modus Ponens* rule for the *material* implication $\neg x\vee y$:

$$\{x,\neg x\vee y\}\vdash y, \tag{4.7}$$

being an extension of NP_{01}. A Kleene algebra is said to be *classical*, provided the quasi-identity $\nabla(4.7)$ is true in it. The prevariety of all classical members of K_3 is denoted by $\mathsf{CK}_3\subseteq\mathsf{NPK}_3$.

Lemma 4.13 (cf. Lemma 4.14 of [12] for the constant-free case with $B=\{\mathsf{f},\mathsf{t}\}$)**.** *Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{DM}_{4[,01]}$ and $\varphi\in\mathrm{Fm}_{\Sigma_{[01]}}$. Suppose $B\cup\{\mathsf{b}\}$ forms a subalgebra of $\mathfrak{DM}_{4[,01]}$. Then,* $(\langle\mathfrak{B},B\cap\{\mathsf{t},\mathsf{b}\}\rangle\in\mathrm{Mod}(\varphi))\Leftrightarrow(\langle\mathfrak{DM}_{4[,01]}{\restriction}(B\cup\{\mathsf{b}\}),(B\cup\{\mathsf{b}\})\cap\{\mathsf{t},\mathsf{b}\}\rangle\in\mathrm{Mod}(\varphi))$.

Proof. The metaimplication from right to left is by the fact $\mathfrak{B}$ is a subalgebra of $\mathfrak{DM}_{4[,01]}{\restriction}(B\cup\{\mathsf{b}\})$. Conversely, assume $\langle\mathfrak{DM}_{4[,01]}{\restriction}(B\cup\{\mathsf{b}\}),(B\cup\{\mathsf{b}\})\cap\{\mathsf{t},\mathsf{b}\}\rangle\notin\mathrm{Mod}(\varphi)$, in which case there exists some $h\in\hom(\mathfrak{Fm}_{\Sigma_{[01]}},\mathfrak{DM}_{4[,01]}{\restriction}(B\cup\{\mathsf{b}\}))$ such that $h(\varphi)\in\{\mathsf{f},\mathsf{n}\}$. Take any $b\in B\neq\varnothing$. Define a mapping g from the set V of all variables to B by setting:

$$g(x)\triangleq\begin{cases}b & \text{if } h(x)=\mathsf{b},\\ h(x) & \text{otherwise,}\end{cases}$$

for all $x\in V$. Let $e\in\hom(\mathfrak{Fm}_{\Sigma_{[01]}},\mathfrak{B})$ extend g. Recall that operations of $\mathfrak{DM}_{4[,01]}$ (and so those of $\mathfrak{DM}_{4[,01]}{\restriction}(B\cup\{\mathsf{b}\})$) are *regular*, i.e., monotonic with respect to the partial ordering $\sqsubseteq$ on 2^2 defined by $(\vec{a}\sqsubseteq\vec{b})\stackrel{\text{def}}{\Longleftrightarrow}((a_0\leqslant b_0)\&(b_1\leqslant a_1))$, for all $\vec{a},\vec{b}\in 2^2$. Moreover, $e(x)=g(x)\sqsubseteq h(x)$, for all $x\in V$, in which case, we have $e(\varphi)\sqsubseteq h(\varphi)\sqsubseteq\mathsf{n}$, and so we eventually get $e(\varphi)\in\{\mathsf{f},\mathsf{n}\}$, as required. □

Proposition 4.14. $MP_{01}=PC_{01}$.

Proof. Clearly, (4.7) is true in $\langle\mathfrak{B}_2,2\setminus 1\rangle$, so $MP_{01}\subseteq PC_{01}$. Conversely, consider any $(\Gamma\vdash\varphi)\in PC_{01}$. Then, by the Compactness Theorem (cf., e.g., [5]) with propositional variables treated as nullary predicates, there is some $\Delta\in\wp_\omega(\Gamma)$ such that $(\Delta \vdash \varphi) \in PC_{01}$. By induction on the cardinality of any $\Xi \in \wp_\omega(\mathrm{Fm}_{\Sigma_{01}})$, let us prove that, for all $\phi \in \mathrm{Fm}_{\Sigma_{01}}$ such that $(\Xi \vdash \phi) \in PC_{01}$, it holds that $(\Xi \vdash \phi) \in MP_{01}$. The case, when $\Xi = \varnothing$, is by Lemma 4.13 with $B = \{\mathsf{f}, \mathsf{t}\}$ and the inclusion $LP_{01} \subseteq MP_{01}$. Otherwise, take any $\psi \in \Xi$, in which case $\Theta \triangleq (\Xi \setminus \{\psi\}) \in \wp_\omega(\mathrm{Fm}_{\Sigma_{01}})$, while $|\Theta| < |\Xi|$, whereas, by the Deduction Theorem for PC_{01} with respect to the material implication, we have $(\Theta \vdash (\neg\psi \vee \phi)) \in PC_{01}$, and so, by the induction hypothesis, we get $(\Theta \vdash (\neg\psi \vee \phi)) \in MP_{01}$. Then, by (4.7), we eventually get $(\Xi \vdash \phi) \in MP_{01}$. Thus, in particular, $(\Delta \vdash \varphi) \in MP_{01}$, in which case $(\Gamma \vdash \varphi) \in MP_{01}$, as required. □

Theorem 4.15. *Proper consistent extensions of LP_{01} form the two-element chain $NP_{01} \subsetneq MP_{01} = PC_{01}$, both having same axioms as LP_{01} does, and so being non-axiomatic.*

Proof. First, by Corollary 4.12, Proposition 4.14 and the fact that (4.6) is not true in $\mathfrak{K}^{\nabla}_{3,01}$ under $[x/1, y/0]$, while (4.7) is not true in $\langle \mathfrak{K}_{3,01}, 3 \setminus 1\rangle \times \langle \mathfrak{B}_2, 2 \setminus 1\rangle$ under $[x/\langle 1,1\rangle, y/\langle 0,1\rangle]$, we conclude that $LP_{01} \subsetneq NP_{01} \subsetneq MP_{01} = PC_{01}$. Next, consider any non-trivial prevariety $\mathsf{P} \subseteq \mathsf{K}_3$ and the following three exhaustive cases:

(1) $\mathsf{P} \subseteq \mathsf{CK}_3$.
Then, by Lemma 4.6, $\mathfrak{B}_2 \in \mathsf{P}$. Hence, by Proposition 4.14, the logic of P^{∇} is equal to PC_{01}.

(2) $\mathsf{P} \not\subseteq \mathsf{CK}_3$ but $\mathsf{P} \subseteq \mathsf{NPK}_3$.
Take any $\mathfrak{A} \in (\mathsf{P} \setminus \mathsf{CK}_3) \neq \varnothing$, in which case there are some $a, b \in A$ such that $\neg^{\mathfrak{A}} a \leqslant^{\mathfrak{A}} a$, $\neg^{\mathfrak{A}} b \not\leqslant^{\mathfrak{A}} b$ and $c \triangleq (b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} c$. Put:

$$\begin{aligned} d &\triangleq (a \wedge^{\mathfrak{A}} c), \\ e &\triangleq (a \wedge^{\mathfrak{A}} b), \\ f &\triangleq (e \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} b). \end{aligned}$$

Using the fact that $\mathfrak{A} \in \mathsf{KA}$, it is routine checking that:

$$d = (\neg^{\mathfrak{A}} d \vee^{\mathfrak{A}} e), \tag{4.8}$$

$$f = (d \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} e). \tag{4.9}$$

Moreover, since $\neg^{\mathfrak{A}} d \leqslant^{\mathfrak{A}} d \leqslant^{\mathfrak{A}} f$, while $e \leqslant^{\mathfrak{A}} b \not\geqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} b$, whereas $\mathfrak{A}$ is both non-paraconsistent and non-one-element, we also have $e \not\geqslant^{\mathfrak{A}} d \not\leqslant^{\mathfrak{A}} \neg^{\mathfrak{A}} d$. Hence, by (4.8) and (4.9), we conclude that the mapping $g : (3 \times 2) \to A$, given by:

$$\begin{aligned} g(\langle 0,0\rangle) &\triangleq \neg^{\mathfrak{A}} f, \\ g(\langle 1,0\rangle) &\triangleq \neg^{\mathfrak{A}} d, \\ g(\langle 2,0\rangle) &\triangleq \neg^{\mathfrak{A}} e, \\ g(\langle 0,1\rangle) &\triangleq e, \\ g(\langle 1,1\rangle) &\triangleq d, \\ g(\langle 2,1\rangle) &\triangleq f, \end{aligned}$$

is an embedding of $(\mathfrak{K}_3 \times \mathfrak{K}_2)$ into $\mathfrak{A}{\restriction}\Sigma$. Consider the following complementary subcases:

(a) $f = \top^{\mathfrak{A}}$,
in which case $\neg^{\mathfrak{A}} f = \bot^{\mathfrak{A}}$, and so g is an embedding of $\mathfrak{K}_{3,01} \times \mathfrak{B}_2$ into $\mathfrak{A}$. Hence, by Proposition 4.11, $\mathsf{P} = \mathsf{NPK}_3$. Therefore, the logic of P^{∇} is equal to NP_{01}.

(b) $f \neq \top^{\mathfrak{A}}$,[5]
in which case $\neg^{\mathfrak{A}} f \neq \bot^{\mathfrak{A}}$. Since $\{1\}$ forms a subalgebra of $\mathfrak{K}_3$, the set $B \triangleq (((3 \times 2) \times \{1\}) \cup \{\langle 0,0,0\rangle, \langle 2,1,2\rangle\})$ forms a subalgebra of $(\mathfrak{K}_{3,01} \times \mathfrak{B}_2) \times \mathfrak{K}_{3,01}$. Then, the mapping $h : B \to A$, given by:

$$\begin{aligned} h(\langle 0,0,0\rangle) &\triangleq \bot^{\mathfrak{A}}, \\ h(\langle 2,1,2\rangle) &\triangleq \top^{\mathfrak{A}}, \\ h(\langle i,j,1\rangle) &\triangleq g(\langle i,j\rangle), \end{aligned}$$

where $i \in 3$ and $j \in 2$, is an embedding of $\mathfrak{B} \triangleq (((\mathfrak{K}_{3,01} \times \mathfrak{B}_2) \times \mathfrak{K}_{3,01}) \restriction B)$ into $\mathfrak{A}$. On the other hand, $\pi_0 \restriction B$ is a strict surjective homomorphism from $\mathfrak{B}^{\nabla}$ onto $(\mathfrak{K}_{3,01} \times \mathfrak{B}_2)^{\nabla}$. Hence, by Corollary 4.12, the logic of P^{∇} is equal to NP_{01}.

Thus, in any subcase, the logic of P^{∇} is equal to NP_{01}.

(3) $\mathsf{P} \not\subseteq \mathsf{NPK}_3$.
Take any $\mathfrak{A} \in (\mathsf{P} \setminus \mathsf{NPK}_3) \neq \varnothing$. Then, $\mathfrak{A}$ is not one-element and there is some $a \in A$ such that $\neg^{\mathfrak{A}} a = a$. In that case, the mapping $e : 3 \to A$, given by:

$$\begin{aligned} e(0) &\triangleq \bot^{\mathfrak{A}}, \\ e(1) &\triangleq a, \\ e(2) &\triangleq \top^{\mathfrak{A}}, \end{aligned}$$

is an embedding of $\mathfrak{K}_{3,01}$ into $\mathfrak{A}$. Hence, $\mathsf{P} = \mathsf{K}_3$. Therefore, the logic of P^{∇} is equal to LP_{01}.

Finally, Lemma 4.13 with $B = \{\mathsf{f}, \mathsf{t}\}$ completes the argument. □

Since the instance $\frac{\varnothing \rightarrowtail y, x; x \rightarrowtail \varnothing}{\varnothing \rightarrowtail y}$ of ortho-Cut is not true in $\mathcal{DM}_{4-\mathsf{b}} \otimes \mathcal{DM}_{4-\mathsf{nb}}$ under $[x/\langle \mathsf{b}, \mathsf{f}\rangle, y/\langle \mathsf{f}, \mathsf{t}\rangle]$, combining Corollaries 4.4, 4.5, 4.12, Lemma 3.1 and Theorem 4.15 with [9], we immediately get:

Corollary 4.16. $\mathbb{LK}_{\mathrm{S[W]C}}$ *are the only proper consistent extensions of* $\mathbb{LK}_{\mathrm{S}}$*, both having same derivable axioms as* $\mathbb{LK}_{\mathrm{S}}$ *does, and so being non-axiomatic. Moreover,* $\mathcal{DM}_{4-\mathsf{b}} \otimes \mathcal{DM}_{4-\mathsf{nb}}$ *is characteristic for* $\mathbb{LK}_{\mathrm{SWC}}$*, in which case some (ortho-)Cuts are not derivable in it, so it is not multiplicative, and so has no characteristic class of crisp two-side* Σ_{01}*-matrices, while* $\mathbb{LK}_{\mathrm{S[O]C}}$ *have same derivable rules.*

It is Corollary 4.16 that justifies the fuzzy semantic approach to sequent calculi with *basic* structural rules developed here.

Since the admissibility of rules of $\mathbb{LK}_{\mathrm{S}}$ in the Cut-free version of LK is quite immediate, Corollary 4.16 provides a new deep insight into Cut Elimination in LK. More precisely, taking the proof of Lemma 4.13 into account, Cut Elimination in LK appears to be just a consequence of the well-known regularity of operations of Belnap's four-valued (more specifically, Kleene' three-valued) logic.

4.2. **Axiomatic extensions of Belnap's logic.**

Theorem 4.17. $LP_{[01]}$ *is the only proper consistent axiomatic extension of* $B_{4[,01]}$ *and is relatively axiomatized by the Excluded Middle Law axiom* $x \vee \neg x$.

Proof. Consider any $\mathcal{A} \subseteq \mathrm{Fm}_{\Sigma_{[01]}}$ such that the axiomatic extension E of $B_{4[,01]}$ relatively axiomatized by $\mathcal{A}$ is both proper and consistent, in which case $\mathcal{A} \neq \varnothing$, while, by Proposition 2.1, the set $\mathsf{S} \triangleq (\mathrm{Mod}(\mathcal{A}) \cap \mathbf{S}_*(\langle \mathfrak{DM}_{4[,01]}, \{\mathsf{t}, \mathsf{b}\}\rangle))$ defining E is not empty and does not contain $\langle \mathfrak{DM}_{4[,01]}, \{\mathsf{t}, \mathsf{b}\}\rangle$. Take any $\mathcal{B} \in \mathsf{S}$,

[5] It is this subcase that is the pecularity of the bounded case making the latter essentially beyond the scopes of [12], and so the present study — beyond [18].

in which case it is both consistent and, as $\mathcal{A} \neq \varnothing$, truth-non-empty. Hence, $\{\mathsf{f}, \mathsf{t}\} \subseteq B$. Therefore, if n was in B, then $B \cup \{\mathsf{b}\}$ would be equal to 2^2, in which case, by Lemma 4.13 with $B = \{\mathsf{f}, \mathsf{n}, \mathsf{t}\}$, $\langle \mathfrak{DM}_{4[,01]}, \{\mathsf{t}, \mathsf{b}\}\rangle$ would belong to S. Thus, $B \in \{\{\mathsf{f}, \mathsf{t}\}, \{\mathsf{f}, \mathsf{b}, \mathsf{t}\}\}$. Then, by Lemma 4.13, we conclude that $\langle \mathfrak{DM}_{4[,01]-\mathsf{n}}, \{\mathsf{t}, \mathsf{b}\}\rangle \in \mathsf{S} \subseteq \mathbf{S}_*(\langle \mathfrak{DM}_{4[,01]-\mathsf{n}}, \{\mathsf{t}, \mathsf{b}\}\rangle)$, and so $E = LP_{[01]}$. Finally, $(\mathrm{Mod}(x \vee \neg x) \cap \mathbf{S}_*(\langle \mathfrak{DM}_{4[,01]}, \{\mathsf{t}, \mathsf{b}\}\rangle)) = \mathbf{S}_*(\langle \mathfrak{DM}_{4[,01]-\mathsf{n}}, \{\mathsf{t}, \mathsf{b}\}\rangle)$. Then, Proposition 2.1 completes the argument. □

This strengthens Corollary 5.3 of [7]. In this way, combining Corollary 4.5 and Theorem 4.17 with [9], we eventually get:

Corollary 4.18. $\mathbb{LK}_{\mathrm{S}}$ *is the only proper consistent axiomatic extension of* $\mathbb{LK}$.

5. Conclusions

The principal methodological contribution of this work consists in proper extending the paradigm "fuzziness versus non-multiplicativity" going back to [13] to two-side sequent calculi with merely *basic* structural rules as well as signed sequent calculi of [15].

And what is more, the present study comprehensively discloses the hidden many-valued substance of substructural varsions of LK, partially discovered in [18] for the Cut-free version. In this connection, it yields a new deep insight into the Cut Elimination in LK appearing to be just a consequence of the well-known regularity of operations of Belnap's four-valued (more specfifically, Kleene's three-valued) logic.

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