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Modular Map of Collatz Sequence and Its Implications

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Abstract

Let $C : \mathbb{N}+1 \rightarrow \mathbb{N}+1$ and $C(n) = \frac{n}{2}$ if n is even and $C(n) = 3n+1$ if n is odd for all $n \in \mathbb{N}+1$. The Collatz Conjecture states that there exists a finite number k such that $C^k(n) = 1$. In this paper, a modular map of Collatz sequence and several implications are presented, which serve as insights into the further analysis of the Collatz Conjecture.

1 Modular Map of Collatz Sequence

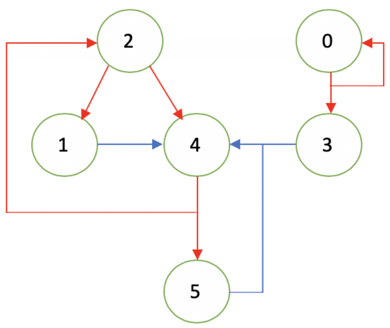

Figure 1: Modular Map of Collatz Sequence

Presented above is the modular map of any Collatz sequence. Every node on the map is a number that represents a specific term in a particular Collatz sequence by its congruence to a number $n \in \mathbb{N}+1, 0 \leq n <$

6 in modulo 6. For example, the Collatz sequence $(7, 22, 11, 34, 17, ...)$ is represented on the modular map as $(1, 4, 5, 4, 5, ...)$.
In the modular map generated, 3 particular loops are identified. Firstly, the α side loop is the finite sequence $(5, 4, 5, 4, ...)$ on the map. Secondly, the β side loop is the finite sequence $(0, 0, 0, 0, ...)$ on the map. Finally, the γ main loop is the sequence that is a combination of the sequences $(2, 4, 2)$ and $(2, 1, 4, 2)$ on the map.
Last but not least, the relationships between the nodes are color-coded. A blue arrow denotes an increasing relationship, in which $C(n) = 3n + 1$. A red arrow denotes a decreasing relationship, in which $C(n) = \frac{n}{2}$.

2 Implications

2.1 Bounded Number of Iterations in the α Side Loop

Lemma 1. *If k is the number of iterations of the first α side loop for a particular $n \equiv 5\,(mod\,6)$, then k must has a finite value.*

Proof. According to Theorem 2 in the article "An Analysis of Collatz Conjecture", for any positive integer n, $C^t(n) = 6z + 2$, with finite $t, z \in \mathbb{N} + 1$[1]. In Lemma 1, although we set a particular rule for n in that it must be congruent to $5\,(mod\,6)$, Theorem 2 in the article "An Analysis of Collatz Conjecture" still holds. From this information, we can infer that the modular sequence for such a number n is $(5, 4, 5, 4, ...5, 4, 2, ...)$, and therefore the α side loops must have been terminated after $k = \frac{t}{2}$ iterations. Since t is a finite positive integer, k must also be a finite positive integer as well. $\square$

Lemma 2. *Let $\Phi(n) = \frac{3n+1}{2}$ be a function that represents a full iteration of the α side loop. If k is the total number of iterations of the first α side loop and $n \equiv 5\,(mod\,6)$, then*

$$(\frac{3}{2})^k(n + 1) - 1 \equiv 2\,(mod\,6)$$

Proof. As demonstrated in the article "An analysis of Collatz Conjecture" [1],

$$\Phi^k(n) = (\frac{3}{2})^k(n + 1) - 1$$

According to the modular map of the Collatz sequence presented in section, the sequence $5 \to 4 \to 5$ represents one full iteration of the α side loop. The operation Φ carries on k full iterations such that the last iteration can be represented as $5 \to 4 \to 2$, which reaches a number that is congruent to $2\,(mod\,6)$. Therefore, we can conclude that if $n \equiv 5\,(mod\,6)$ and the corresponding α side loop iterates k full iterations, then

$$(\frac{3}{2})^k(n+1) - 1 \equiv 2\,(mod\,6)$$

□

Through Lemma 1 and Lemma 2, we can conclude that the α side loop iterates finitely many times and achieves a finitely bounded value.

2.2 Special Cases Where n is a Prime Number

Lemma 3. *If n is a prime, and its first α side loop in its modular map iterates k number of times, then*

$$(\frac{3}{2})^{k+1}(n+1) - 1 \equiv 2\,(mod\,6)$$

Proof. Let n be any prime number such that $n > 3$. Because every prime can either be written in the form $6p-1$ or $6p+1$, with $p \in \mathbb{N}+1$, n must be starting from node 1 or 5 on the modular map of Collatz sequence in section 1. Let us examine both cases as follows.
In the case that $n \equiv 1\,(mod\,6)$, $\frac{3n+1}{2}$ is the first term of the first α side loop of its sequence, which represents the progression $1 \to 4 \to 5$. In fact, this progression can be treated as an additional iteration of Φ with n as the input, apart from the full iterations of the corresponding α side loop. After the α side loop has fully iterated k number of times, the sequence reaches a term that is congruent to $2(mod 6)$. As a result, if n is a prime number and $n \equiv 1\,(mod\,6)$, then

$$(\frac{3}{2})^{k+1}(n+1) - 1 \equiv 2(mod 6) \qquad (1)$$

Similarly, in the case that $n \equiv 5(mod6)$, $3n+1$ is the first term of the first α side loop of its sequence. Applying the same logical reasoning as above, we can say that if n is a prime number and $n \equiv 5\,(mod\,6)$, then

$$(\frac{3}{2})^{k+1}(n+1) - 1 \equiv 2\,(mod\,6) \qquad (2)$$

Finally, it is observed that the results gathered in (1) and (2) are identical, which allow us to make the conclusion that for any prime number $n > 3$,

$$(\frac{3}{2})^{k+1}(n+1) - 1 \equiv 2\,(mod\,6)$$

□

2.3 Special Cases Where n is an Odd Multiple of 3

Lemma 4. *If n is an odd multiple of 3, and its first α side loop in its modular map iterates k number of times, then*

$$(\frac{3}{2})^{k+1}(n+1)-1 \equiv 2\,(mod\,6)$$

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Proof. It is clear that any odd multiple of 3 is congruent to $3\,(mod\,6)$. Based on the modular map of Collatz sequence in section 1, in which the progression $3 \to 4 \to 5$ is similar to the progression $1 \to 4 \to 5$ and $5 \to 4 \to 5$, we can apply the same reasoning from the proof of Lemma 3 into this proof in that the progression $3 \to 4 \to 5$ represents an additional iteration of Φ and say that if n is an odd multiple of 3, then

$$(\frac{3}{2})^{k+1}(n+1)-1 \equiv 2\,(mod\,6)$$

□

Lemma 5. *Let n be an odd multiple of 3 and x be the first number that is congruent to $2(mod6)$ such that $C^k : n \to x$ for some $k \in \mathbb{N}+1$, then x is unique for every n.*

Proof. Let n be an odd multiple of 3 and S be the sequence of its modular map. Based on the modular map of Collatz sequence in section 1,

$$S = (3, 4, 5, 4, 5, ..., 4, 5, 2, ...)$$

Because node 4, considering only the α and β side loops, can either be reached from node 3 or node 5, if we start from $x = S_{2k+2}$ and deduce backwards, the α side loop will be exited after k full iterations and the β side loop is immediately entered, leading us to node 3. From here, we can only reach node 0 if we continue to go backwards, and thus we can conclude that x is unique for n. □

Lemma 6. *Let n be a number that is congruent to $2\,(mod\,6)$ and x be the first odd multiple of 3 such that $C^{-k} : n \to x$, in which C^{-1} is the inverse of C, for some $k \in \mathbb{N}+1$, then x is not necessarily unique for every n.*

Proof. In this proof, we are moving in the reverse direction of the arrows in Figure 1. If node 2 is preceded by an α side loop that is in turn preceded by node 1, then this particular α side loop cannot be entered from node 3, as an odd number simply cannot be congruent to 1 $(mod\, 6)$ and 3 $(mod\, 6)$ at the same time. Therefore, to reach x, we must exit this α side loop, past another number that is congruent to 2 $(mod\, 6)$, denoted n', and enter another α side loop. If this α side loop is preceded by node 3, then n and n' share the same x, with x being the first odd multiple of 3 if we map the arrows backwards from either n or n'. If this α side loop is not preceded by node 3, then we repeat the process of mapping backwards, and as a result, many numbers that are congruent to 2 $(mod\, 6)$ will share the same x, or not reaching such a number x at all. Thus, we can conclude that x is not necessarily unique for every n, with $n \equiv 2\ (mod\, 6)$. □

Lemma 7. *Every number $n \equiv 3\ (mod\, 6)$ must be able to be mapped forwards to a distinct number $x \equiv 2\ (mod\, 6)$, but every number $x \equiv 2 (mod 6)$ is not necessarily be able to be mapped backwards to a distinct number $n \equiv 3\ (mod\, 6)$.*

Proof. Lemma 7 naturally flows from lemma 5 and lemma 6. □

References

[1] Francis Charles Motta, Henrique Roscoe de Oliveira, Thiago Aparecido Catalan, and Eduardo Gueron. An analysis of the collatz conjecture.