



## New Bound on the Chebyshev Function and the Riemann Hypothesis

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# New Bound on the Chebyshev function and the Riemann Hypothesis

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## Abstract

Under the assumption that the Riemann hypothesis is true, von Koch deduced the improved asymptotic formula  $\theta(x) = x + O(\sqrt{x} \times \log^2 x)$ , where  $\theta(x)$  is the Chebyshev function. We prove if there exists some real number  $x \geq 10^8$  such that  $\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x$ , then the Riemann hypothesis should be false. In this way, we show that under the assumption that the Riemann hypothesis is true, then  $\theta(x) < x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x$  for all  $x \geq 10^8$ .

*Keywords:* Riemann hypothesis, Nicolas inequality, Chebyshev function, prime numbers  
*2000 MSC:* 11M26, 11A41, 11A25

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## 1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$  [1]. The Riemann hypothesis belongs to the David Hilbert's list of 23 unsolved problems [1]. Besides, it is one of the Clay Mathematics Institute's Millennium Prize Problems [1]. This problem has remained unsolved for many years [1]. In mathematics, the Chebyshev function  $\theta(x)$  is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where  $p \leq x$  means all the prime numbers  $p$  that are less than or equal to  $x$ . Say Nicolas( $p_n$ ) holds provided

$$\prod_{q \leq p_n} \frac{q}{q-1} > e^\gamma \times \log \theta(p_n).$$

The constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant,  $\log$  is the natural logarithm, and  $p_n$  is the  $n^{\text{th}}$  prime number. The importance of this property is:

**Theorem 1.1.** [2], [3]. Nicolas( $p_n$ ) holds for all prime numbers  $p_n > 2$  if and only if the Riemann hypothesis is true.

We know the following properties for the Chebyshev function:

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**Theorem 1.2.** [4]. *If the Riemann hypothesis holds, then*

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x)$$

for all  $x \geq 10^8$ .

**Theorem 1.3.** [5]. *For  $2 \leq x \leq 10^8$*

$$\theta(x) < x.$$

We also know that

**Theorem 1.4.** [6]. *If the Riemann hypothesis holds, then*

$$\left( \frac{e^{-\gamma}}{\log x} \times \prod_{q \leq x} \frac{q}{q-1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}$$

for all numbers  $x \geq 13.1$ .

Let's define  $H = \gamma - B$  such that  $B \approx 0.2614972128$  is the Meissel-Mertens constant [7]. We know from the constant  $H$ , the following formula:

**Theorem 1.5.** [8].

$$\sum_q \left( \log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = \gamma - B = H.$$

For  $x \geq 2$ , the function  $u(x)$  is defined as follows

$$u(x) = \sum_{q > x} \left( \log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right).$$

We use the following theorems:

**Theorem 1.6.** [9]. *For  $x > -1$ :*

$$\frac{x}{x+1} \leq \log(1+x).$$

**Theorem 1.7.** [10]. *For  $x \geq 1$ :*

$$\log\left(1 + \frac{1}{x}\right) < \frac{1}{x+0.4}.$$

Let's define:

$$\delta(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

**Definition 1.8.** *We define another function:*

$$\varpi(x) = \left( \sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality  $\varpi(x) > u(x)$  is satisfied for a number  $x \geq 3$  if and only if  $\text{Nicolas}(p)$  holds, where  $p$  is the greatest prime number such that  $p \leq x$ . In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.

## 2. Results

**Theorem 2.1.** *The Riemann hypothesis is true if and only if the inequality  $\varpi(x) > u(x)$  is satisfied for all numbers  $x \geq 3$ .*

*Proof.* In the paper [3] is defined the function:

$$f(x) = e^\gamma \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q-1}{q}.$$

We know that  $f(x)$  is lesser than 1 when Nicolas( $p$ ) holds, where  $p$  is the greatest prime number such that  $2 < p \leq x$ . In the same paper, we found that

$$\log f(x) = U(x) + u(x)$$

where  $U(x) = -\varpi(x)$  [3]. When  $f(x)$  is lesser than 1, then  $\log f(x) < 0$ . Consequently, we obtain that

$$-\varpi(x) + u(x) < 0$$

which is the same as  $\varpi(x) > u(x)$ . Therefore, this is a consequence of the theorem 1.1.  $\square$

**Theorem 2.2.** *If the Riemann hypothesis holds, then*

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > 1$$

for all numbers  $x \geq 13.1$ .

*Proof.* Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right)$$

after of distributing the terms based on the theorem 1.4 for all numbers  $x \geq 13.1$ . If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q \leq x} \log\left(\frac{q}{q-1}\right) < \gamma + \log \log x + \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right).$$

That would be equivalent to

$$\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q}\right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

where we know that

$$\begin{aligned} \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right) &< \frac{1}{\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} + 0.4} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} \end{aligned}$$

according to theorem 1.7 since  $\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} \geq 1$  for all numbers  $x \geq 13.1$ . We use the theorem 1.5 to show that

$$\sum_{q \leq x} \left( \log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = H - u(x)$$

and  $\gamma = H + B$ . So,

$$H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.$$

We eliminate the value of  $H$  and thus,

$$-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is equal to

$$u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Under the assumption that the Riemann hypothesis is true, we know from the theorem 2.1 that  $\varpi(x) > u(x)$  for all numbers  $x \geq 13.1$  and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$

Suppose that  $\theta(x) = \epsilon \times x$  for some constant  $\epsilon > 1$ . Then,

$$\begin{aligned} \log \log \theta(x) - \log \log x &= \log \log(\epsilon \times x) - \log \log x \\ &= \log(\log x + \log \epsilon) - \log \log x \\ &= \log\left(\log x \times \left(1 + \frac{\log \epsilon}{\log x}\right)\right) - \log \log x \\ &= \log \log x + \log\left(1 + \frac{\log \epsilon}{\log x}\right) - \log \log x \\ &= \log\left(1 + \frac{\log \epsilon}{\log x}\right). \end{aligned}$$

In addition, we know that

$$\log\left(1 + \frac{\log \epsilon}{\log x}\right) \geq \frac{\log \epsilon}{\log \theta(x)}$$

using the theorem 1.6 since  $\frac{\log \epsilon}{\log x} > -1$  when  $\epsilon > 1$ . Certainly, we will have that

$$\log\left(1 + \frac{\log \epsilon}{\log x}\right) \geq \frac{\frac{\log \epsilon}{\log x}}{\frac{\log \epsilon}{\log x} + 1} = \frac{\log \epsilon}{\log \epsilon + \log x} = \frac{\log \epsilon}{\log \theta(x)}.$$

Thus,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.$$

If we add the following value of  $\frac{\log x}{\log \theta(x)}$  to the both sides of the inequality, then

$$\begin{aligned} \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} &> \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} \\ &= \frac{\log \epsilon + \log x}{\log \theta(x)} \\ &= \frac{\log \theta(x)}{\log \theta(x)} \\ &= 1. \end{aligned}$$

We know this inequality is satisfied when  $0 < \epsilon \leq 1$  since we would obtain that  $\frac{\log x}{\log \theta(x)} \geq 1$ . Therefore, the proof is done.  $\square$

**Theorem 2.3.** *If there exists some real number  $x \geq 10^8$  such that*

$$\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x,$$

*then the Riemann hypothesis is false.*

*Proof.* If the Riemann hypothesis holds, then

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x)$$

for all  $x \geq 10^8$  due to the theorem 1.2. Now, suppose there is a real number  $x \geq 10^8$  such that  $\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x$ . That would be equivalent to

$$\log \theta(x) > \log\left(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x\right)$$

and so,

$$\frac{1}{\log \theta(x)} < \frac{1}{\log\left(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x\right)}$$

for all numbers  $x \geq 10^8$ . Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log\left(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x\right)}.$$

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log\left(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x\right)} > 1$$

for those values of  $x$  that complies with

$$\theta(x) > x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x$$

due to the theorem 2.2. By contraposition, if there exists some number  $y \geq 10^8$  such that for all  $x \geq y$  the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x)} \leq 1$$

is satisfied, then the Riemann hypothesis should be false. Let's define the function

$$v(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x)} - 1.$$

The Riemann hypothesis is false when there exists some number  $y \geq 10^8$  such that for all  $x \geq y$  the inequality  $v(x) \leq 0$  is always satisfied. We ignore when  $2 \leq x \leq 10^8$  since  $\theta(x) < x$  according to the theorem 1.3. We know that the function  $v(x)$  is monotonically decreasing for every number  $x \geq 10^8$ . The derivative of  $v(x)$  is negative for all  $x \geq 10^8$ . Indeed, a function  $v(x)$  of a real variable  $x$  is monotonically decreasing in some interval if the derivative of  $v(x)$  is lesser than zero and the function  $v(x)$  is continuous over that interval [11]. It is enough to find a value of  $y \geq 10^8$  such that  $v(y) \leq 0$  since for all  $x \geq y$  we would have that  $v(x) \leq v(y) \leq 0$ , because of  $v(x)$  is monotonically decreasing. We found the value  $y = 10^8$  complies with  $v(y) \leq 0$ . In this way, we obtain that  $v(x) \leq 0$  for every number  $x \geq 10^8$ . Hence, the proof is complete.  $\square$

**Theorem 2.4.** *Under the assumption that the Riemann hypothesis is true, then*

$$\theta(x) < x + \frac{1}{\log \log \log x} \times \sqrt{x} \times \log^2 x$$

for all  $x \geq 10^8$ .

*Proof.* This is a direct consequence of the theorem 2.3.  $\square$

## Appendix

We found the derivative of  $v(x)$  in the web site <https://www.wolframalpha.com/input>. Besides, we determine the sign of the function  $v(x)$  using the tool *gp* from the web site <https://pari.math.u-bordeaux.fr>. In the project PARI/GP, the method *sign(F(X))* returns  $-1$  when the function  $F(X)$  is negative in the value of  $X$ . We checked that is negative for  $X = 10^8$  with a real precision of 1000016 significant digits when  $F(X) = v(x)$ . We also checked that is still negative for  $X = 100000!$ , where  $(..)!$  means the factorial function.

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