Fluid Motion in Cracked Porous Media

Ilgar Safarli, Leyla Nasirova and Konul Abdullayeva

February 11, 2025

FLUID MOTION IN CRACKED POROUS MEDIA

Safarli Ilgar
Associate Professor of the Department of Mathematical Analysis and Differential Equations
Sumgayit State University,
Sumgayit, Azerbaijan
ilqar.safarli@sdu.edu.az

Nasirova Leyla
Teacher of Mathematical Analysis and Differential Equations
Sumgayit State University,
Sumgayit, Azerbaijan
leyla.nasirova@sdu.edu.az

Abdullayeva Konul
Teacher of Mathematical Analysis and Differential Equations
Sumgayit State University,
Sumgayit, Azerbaijan
konul.abdullayeva@sdu.edu.az

Abstract - The paper describes nonstationary problems of thermodynamical and hydrodynamical processes whose mass change is used in heterogeneous media for an arbitrary geometry of thermal, filtrational or diffusive flow described in appropriate terminologies by the system of parabolic type differential equations.

***Keywords:** thermodynamical process, hydrodynamical process, Laplace transforms, integral transformation, nonstationary problems, functional dependence.*

I. INTRODUCTION.

Using the Laplace integral transformation in time variable, its main properties and the ideas of expansion of fractional rational functions in infinite series by the roots of their denominators, we suggest methods for constructing exact solutions of problems of nonstationary processes in heterogeneous continuum described by the system of Barenblatt-Zheltov differential equations. Applicability of the obtained methods is shown on solutions of specific problems of thermodynamics and hydrodynamics in these media.

II. STATEMENT AND FORMULATION OF THE PROBLEM.

Nonstationary problems of thermodynamical and hydrodynamical processes of mass exchange in heterogeneous media for arbitrary geometry of thermal, filtrational or diffusive flow are described in appropriate terminologies by the system of parabolic type differential equations

$$\frac{1}{k_0}\frac{\partial^2\phi_1}{\partial\xi^2}+\frac{\sigma}{\xi}\frac{\partial\phi_1}{\partial\xi}+\lambda(\phi_2-\phi_1)-(1-\omega)\frac{\partial\phi_1}{\partial\tau}=0, \quad (1)$$

$$\frac{\partial^2\phi_2}{\partial\xi^2}+\frac{\sigma}{\xi}\frac{\partial\phi_2}{\partial\xi}+\lambda(\phi_1-\phi_2)-\omega\frac{\partial\phi_2}{\partial\tau}=0,$$

where $\phi_i(\xi;\tau), (i=1,2)$ are the desired generalized functions that describe the change of the process at arbitrary points ξ of mutually connected media 1 and 2 at arbitrary moment of time τ; ω and λ are constant parameters [1], that characterize the medium under consideration, the values $\sigma=0,1,2$ characterize the flow geometry and correspond to linear, plane radial and centrally-symmetric change of the process

$$\omega=\frac{\beta_2^*}{\beta_1^*+\beta_2^*}, 1-\omega=\frac{\beta_1^*}{\beta_1^*+\beta_2^*}$$

Substituting the expression of the function $\phi_1(\xi,\tau)$ from the second equation to the first one and neglecting the quantity $\frac{1}{\alpha R_c^2}\nabla^4\phi_2$, we can represent the given system in the form (index 2 is omitted)

$$\frac{1+k_0}{k_0}\nabla^2\phi+\frac{\frac{\omega}{k_0}+1-\omega}{\lambda}\frac{\partial}{\partial\tau}(\nabla^2\phi)-$$

$$-\frac{\omega(1-\omega)}{\lambda}\frac{\partial^2\phi}{\partial\tau^2}-\frac{\partial\phi}{\partial\tau}=\frac{k_0}{\lambda}\nabla^4\phi, \quad (2)$$

$$\nabla^2=\frac{\partial^2}{\partial\xi^2}+\frac{\sigma}{\xi}\frac{\partial}{\partial\xi}.$$

Many native and foreign researchers are engaged in the solution of important thermodynamical, hydrodynamical and gasdynamical problems for the given system of equations, the results of which are in the authors' monographs [1-5]. It should be noted that application of integral transformations to specific problems for the system (1) leads to mathematical obstacles difficult to overcome and the researcher use simplified special cases [6] to get over them. Such an approach leads to approximate solutions of the considered problems suitable for limited interval of change of variables and further become a subject of discussions.

III. MAIN PART.

Application of Laplace integral transform with a transformation parameter S both to the system (1) and to the equation (2) lead them to the second order differential equation

$$\frac{1+k_0}{k_0}\nabla^2\psi+\frac{1-\omega+\frac{\omega}{k_0}}{\lambda}\nabla[s\psi-\psi(\xi;0)]-$$

$$-\frac{\omega(1-\omega)}{\lambda}\left[s^2\psi-s\psi(\xi;0)-\frac{\partial}{\partial\tau}\psi(\xi;0)\right]-$$

$$-s\psi+\psi(\xi;0)=0. \quad (3)$$

whose general solution with, regard to the initial condition

$$\psi_i(\xi,s)=A_i\xi^\nu I_\nu\left(\xi\sqrt{S(s)}\right)+B_i\xi^\nu K_\nu\left(\xi\sqrt{S(s)}\right), \quad (4)$$

$$\nu=\frac{1-\sigma}{2}, S(s)=s\frac{\omega(1-\omega)s+\lambda}{\left(1-\omega+\frac{\omega}{k_0}\right)s+\lambda\frac{2+k_0}{k_0}}. \quad (5)$$

$I_\nu(z)$ and $K_\nu(z)$ are Bessel functions from the first and second kinds imaginary argument of order ν, A and B are integration constants dependent on the transformation parameter.

For $\omega=1$ or $\lambda\to\infty$ equations (1) and (2) pass to the known Fourier equation

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\sigma}{\xi}\frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \tau} = 0, (\sigma = 0; 1; 2) \qquad (6)$$

and from (4) due to the validity of the relation

$$\lim_{k_0 \to \infty} S(s) = s \ \text{ for } \omega = 1. \qquad (7)$$

we get a general solution of (6) of the form

$$\psi(\xi, \nu) = A\xi^{\nu} I_{\nu}(\xi\sqrt{s}) + B\xi^{\nu} K_{\nu}(\xi\sqrt{s}). \qquad (8)$$

Comparison of solutions of (5) and (8) show that they differ between themselves only by functional dependence relative to the transformation parameter S in the arguments of functions included in these general solutions. Hence it follows that for the known solution of the equation (6) in the Laplace transformation under the certain boundary conditions

$$[\psi(\xi, s)]_{\substack{\omega=1 \\ \lambda=\infty}} = \phi(s) F(\xi; s) \qquad (9)$$

the appropriate solution of equations (1) and (2) corresponding to it, under the same boundary conditions can be represented in the form

$$\psi(\xi, s) = \phi(s) F[\xi, S(s)], \qquad (10)$$

where $\phi(s)$ is the Laplace transform of a function given on the medium impact boundary?

Depending on the features of the function $F[\xi, S(s)]$ below we give methods for passing from the image (10) to the original.

Application of generalized A.M.Efros multiplication theorem. Due to complicated functional dependence of S on s for realizing the passage from the images of the form (10) to their originals it is purposeful to use generalized A.M.Erfos multiplication theorem [7], saying (briefly) that

$$\phi(s) F[\xi; S(s)] \doteqdot \int_0^{\infty} f_*(\xi; \theta) g(\tau; \theta) d\theta \qquad (11)$$

where

$$f_*(\xi, \tau) \doteqdot F(\xi, s) \equiv \frac{1}{\phi(s)} [\psi(\xi, s)]_{\omega=1}, \qquad (12)$$

$$g(\tau, \theta) = \phi(s) \, exp[-\theta S(s)]. \qquad (13)$$

To apply formulas (11) to the image (10) we represent in the form

$$\psi(\xi, s) = sG(s)\phi(s) \frac{F[\xi; S(s)]}{S(s)}, \qquad (14)$$

$$G(s) \equiv \frac{S(s)}{s}. \qquad (15)$$

Let

$$f(\xi; \tau) \doteqdot \frac{1}{s} F(\xi; s) \qquad (16)$$

be the corresponding known solution of the problem for the Fourier equation (6). Then for finding the function $g(\tau; \theta)$ due to (13), we represent it allowing for (14) for $\phi(s) = \frac{1}{s}$ (given constant consumption for $\xi = 1$) in the form

$$g(\tau, \theta) \doteqdot G(s) \, exp[-\theta S(s)] =$$

$$= \left(\omega\varepsilon_2 + \varepsilon_2 \frac{\Lambda_1 - \omega\varepsilon_1\Lambda_2}{s + \varepsilon_1\Lambda_2}\right) \times$$

$$\times \exp\,[-\omega\varepsilon_2 s\theta - \Lambda_2(1 - \varepsilon_1\varepsilon_2\omega)\theta +$$

$$+\varepsilon_1\Lambda_2(1 - \omega\varepsilon_1\varepsilon_2)\frac{\theta}{s+\varepsilon_1\Lambda_2}], \qquad (17)$$

$$\varepsilon_1 = \frac{1 + k_0}{k_0}, \varepsilon_2 = \frac{1 - \omega}{1 - \omega + \dfrac{\omega}{k_0}},$$

$$\Lambda_2 = \frac{\lambda}{1 - \omega + \dfrac{\omega}{k_0}}, \Lambda_1 = \frac{\lambda}{1 - \omega}.$$

Realizing the passage to the original in (17) and substituting the found expression in (11), we can represent the solution of the problem in the form

$$\phi_2(\xi; \tau) = \varepsilon_2 \, exp(-\varepsilon_1\Lambda_2\tau) \times$$

$$\times \int_0^{\infty} f(\xi; \theta) \, exp(-c\theta) \times$$

$$\times \left\{\omega\sqrt{\frac{\varepsilon_1 c\theta}{\tau}} I_1\left(2\sqrt{\varepsilon_1 c\theta\tau}\right) + \omega + BI_0\left(2\sqrt{\varepsilon_1 c\theta\tau}\right)\right\} \times$$

$$\times H(\tau - \omega\varepsilon_2\theta) d\theta \qquad (18)$$

$$B = \Lambda_1 - \omega\varepsilon_1\Lambda_2, c = \Lambda_2(1 - \varepsilon_1\varepsilon_2\omega),$$

$$H(\tau - x) = \begin{cases} 0, \tau < x \\ 1, \tau > x \end{cases}. \qquad (19)$$

Formula (18) is the exact solution of the problem for a heterogeneous medium, where the exact solution for the homogeneous medium $\omega = 1, k_0 = \infty$was used.

Accepting other expressions for the functions $G(s)$ and $\phi(s)$, from (10) we can find various analytic expressions for the function $g(\tau; \theta)$ and by the same taken various forms of exact solutions to the problem. So, for example, accepting

$$f_*(\xi, \tau) = F(\xi, s), G(s) = \phi(s) = \frac{1}{s} \qquad (20)$$

we must find the original of the image

$$g(\tau, \theta) \doteqdot \frac{1}{s} exp[-\theta S(s)] \doteqdot \eta(\tau - \omega\theta) \, exp(-\Lambda\theta) +$$

$$+e^{-\Lambda\theta} \int_0^{(\tau-\omega\theta)\eta(\tau-\omega\theta)} exp(-\Lambda z) \, I_1[Z(z)] \sqrt{\frac{\lambda\Lambda}{z}} dz, \qquad (21)$$

$$\eta(\tau - x) = \begin{cases} 1, \tau < x \\ 0, \tau > x \end{cases}, Z(z) = 2\lambda\sqrt{\frac{z}{1 - \omega}}.$$

Allowing for (21) and (11) we can obtain the following exact solution

$$\psi(\xi; \tau) = \int_0^{\tau/\omega} f_\theta \ (\xi; \theta) \, exp(-\Lambda\theta) \, d\theta +$$

$$+\frac{\lambda}{\sqrt{1 - \omega}} \int_0^{\tau/\omega} f_\theta \ (\xi; \theta) \, exp(-\Lambda\theta) \, d\theta \times$$

$$\times \int_0^{(\tau-\omega\theta)\eta(\tau-\omega\theta)} exp(-\Lambda z) \, I_1[Z(z)] \frac{dz}{\sqrt{z}}, \qquad (22)$$

where we used time derivative of the exact solution for a homogeneous medium. In our previous papers we have obtained another form of formulas (18) and (22), more exactly

$$\psi(\xi; \tau) = f\left(\xi; \frac{\tau}{\omega}\right) - \lambda \times$$

$$\times \int_0^{\frac{\tau}{\omega}\eta(\tau-\omega\theta)} f_\theta\ (\xi;\theta)\, exp\big(-\Lambda(\tau-\omega\theta)\big)\, d\theta \times$$

$$\times \int_0^\theta exp(-\lambda z)\, I_0[Z(z)]dz \qquad (23)$$

taking into account both $f(\xi;\theta)$ and $f_\theta^{/}(\xi;\theta)$.

IV. CONCLUSION.

All three formulas (18), (22) and (23) are equivalent and satisfy the zero-initial condition. To illustrate that they satisfy also boundary conditions, it is necessary to consider a specific problem from theory of hydrodynamics.

Example 1. A heterogeneous circular cross section cylinder with a rather long length is bounded from interior with the radius $r = R_c$, on which constant heat flow q is given for zero initial temperature. It is required to determine distribution of temperature in time at an arbitrary point of the body.

In terms of hydrodynamics, the formulated problem corresponds to the development of cracked-porous formation of rather long length in the unresolved state with a well performed according the degree of opening at a constant flow rate.

The solution of the corresponding problem for a homogeneous medium in the Laplace transform has the form:

$$f(\xi;\tau) \doteq \frac{K_0(\xi\sqrt{s})}{S(s)K_1(\sqrt{s})} \equiv \frac{1}{s}F(\xi;s) =$$

$$= \frac{1}{2}\int_0^\infty \frac{J_1(u)Y_0(\xi u) - Y_1(u)J_0(\xi u)}{J_1^2(u) + Y_1^2(u)} \times$$

$$\times [1 - exp(-u^2\tau)]\frac{du}{u^2} \qquad (24)$$

$$\xi = \frac{r}{R_c}, \tau = \frac{xt}{R_c^2},$$

$$f(\xi;\tau) = \frac{2\pi kh}{\mu q}[P_0 - P(\xi,\tau)] \qquad (25)$$

$J_n(x), Y_n(x)$ is the Bessel function of the first and second kinds real argument of order n, P_0 and $P(\xi,\tau)$ is the initial and current pressure in the formation, x is a piezo conductivity coefficient, r and t is a radial coordinate and time, respectively.

Using the formulas (18) and (24) corresponding solution for a cracked-porous formation in denotations

$$\omega = \frac{\beta_2^*}{\beta_1^*+\beta_2^*}, \lambda = \alpha R_c^2\frac{k_1}{k_2}, \tau = \frac{k_2 t}{\mu R_c^2(\beta_1^*+\beta_2^*)} \qquad (26)$$

it is easy to represent in the form

$$\psi(\xi;\tau) = \frac{2}{\pi}\times$$

$$\times \int_0^{\frac{\tau}{\omega}} \{\omega + \lambda I_0[u(\tau,\theta)] + v(\tau,\theta)I_1[u(\tau,\theta)]\} \times$$

$$\times exp[-\Lambda\tau - (1-\xi\omega)\Lambda\theta]d\theta \times$$

$$\times \int_0^\infty \frac{J_1(u)Y_0(\xi u) - Y_1(u)J_0(\xi u)}{J_1^2(u)+Y_1^2(u)}[1 - exp(-u^2\theta)]\frac{du}{u^2}. \quad (27)$$

Example 2. Retain the conditions of the previous problem having accepted $\sigma = 2$ the central symmetric flow and replace the condition of unboundedness of the formation by the condition of the availability of a feed contour on the external semi-spherical surface of the formation of radius $\xi = \xi_0$.

The solution of the given problem for a homogeneous medium has the form

$$\psi(\xi,\tau) \doteq \frac{1}{\xi S}\frac{sh(\xi_0-\xi)\sqrt{S}}{\sqrt{S(s)}ch(\xi_0-1)\sqrt{s} + sh(\xi_0-1)\sqrt{S}} \equiv$$

$$\equiv \frac{1}{\xi S}F(\xi,s) \doteq \frac{\xi_0-\xi}{\xi_0-1} + \frac{2}{\xi}(\xi_0-1)^2 \times$$

$$\times \sum_{m=1}^\infty \frac{1}{\alpha_m \cos\alpha_m}\cdot\frac{\sin\alpha_m\frac{\xi_0-\xi}{\xi_0-1}}{\alpha_m^2+\xi_0(\xi_0-1)} exp\left[-\left(\frac{\alpha_m}{\xi_0-1}\right)^2\tau\right], \quad (28)$$

whereα_m are the roots of the equation?

$$tg\alpha + \alpha(\xi_0-1)^{-1} = 0. \qquad (29)$$

Consequently, according to (16) from (28) we can write out the appropriate Laplace transform for a heterogeneous medium.

$$\psi(\xi,S) \doteq \frac{1}{\xi S}\cdot\frac{sh(\xi_0-\xi)\sqrt{S(s)}}{\sqrt{S(s)}ch(\xi_0-1)\sqrt{S}+sh(\xi_0-1)\sqrt{S(s)}}. \qquad (30)$$

The following expression is the original of the image (30) according to (28)

$$\psi(\xi,\tau) = \frac{2}{\xi}\sum_{m=1}^\infty \alpha_m \frac{\sin\alpha_m\frac{\xi_0-\xi}{\xi_0-1}}{[\alpha_m^2+\xi_0(\xi_0-1)]\cos\alpha_m}. \qquad (31)$$

REFERENCES

[1]. Barenblatt G.I., Entov V.M., Ryzhik V.M. Theory of nonstationary filtration of fluid and gas. «Nedra» publ., M., 1972.

[2]. Ban A., Basniev K.S., Nikolaevskii V.N. On basic filtration equation in compressible porous media. Gostekhizdat, M., 1962.

[3]. Rom E.S. Filtration properties of cracked rocks. «Nedra» publ., M., 1966.

[4]. Zheltov Yu.P. Deformation of rocks. «Nedra» publ., M., 1966.

[5]. Nikolaevskii V.N. and others. Mechanics of saturated porous media. «Nedra» publ., M., 1970.

[6]. Boyarchuk V.T., Donpov K.M. On curves of pressure restoration in the well of a cracked porous collector. PMITF, M., 1971, №5.

[7]. Lavrent'ev M.A., Shabat V.V. Methods of theory of complex variable functions. «Nauka» publ, M., 1973.