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The benefits provided by special perturbation methods based on Hansen's ideal frame are broadly recognized these days, and its use is widely encouraged. However, the origin of these advantages is not always clearly understood, probably due to the mixture of facets to consider in the implementation of an orbit propagator. The increased performance is commonly justified by general statements taken from the old astronomical literature. For instance Brown and Shook's [1] introduction of Hansen's ideal frames as a means to achieve a measure of independence of the in-plane motion from the rotation of the referential, is simply adopted in [2], and, from there, repeated in [3]. Also, the fact that the ideal frame formulation yields the same velocity in the inertial frame and in the ideal, rotating frame is commonly stated as an advantage without giving detailed explanations [4, 5].

However, the origin of the advantages of ideal frames over other frames associated to the orbital plane, as the nodal or apsidal (perifocal) frames is quite clear. They stem from the simplified form of the variation equations this clever setting provides, which releases them from the appearance of inertia terms. This feature was shown using Cartan's exterior calculus by Deprit [6]. Here, we provide an alternative proof without need of resorting to differential geometry.

In an inertial frame $(O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, perturbed Kepler motion is generally described by the differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{X}, \qquad \dot{\boldsymbol{X}} = -(\mu/r^3)\boldsymbol{x} + \boldsymbol{P}, \tag{1}$$

where $\boldsymbol{x}$ is position, $\boldsymbol{X}$ velocity, $r = \|\boldsymbol{x}\|$, μ is the gravitational parameter, $\boldsymbol{P}$ is the disturbing acceleration of the Kepler motion, which may generally depend on position, velocity and time, and overdots denote time differentiation. The angular momentum vector $\boldsymbol{G} = \boldsymbol{x} \times \boldsymbol{X}$, and the eccentricity vector $\boldsymbol{e} = (1/\mu)\boldsymbol{X} \times \boldsymbol{G} - \boldsymbol{x}/r$, are fundamental invariants of the Kepler problem.

The unit vectors $\boldsymbol{u} = \boldsymbol{x}/r$, $\boldsymbol{n} = \boldsymbol{G}/\Theta$, with $\Theta = \|\boldsymbol{G}\|$, and $\boldsymbol{v} = \boldsymbol{n} \times \boldsymbol{u}$, define the orbital frame $(O, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{n})$. In this frame

$$\boldsymbol{x} = r\boldsymbol{u}, \tag{2}$$

and hence $\boldsymbol{X} = \dot{r}\boldsymbol{u} + r\dot{\boldsymbol{u}}$, from which $\boldsymbol{G} = r^2\boldsymbol{u} \times \dot{\boldsymbol{u}}$. Then, $\boldsymbol{G} \times \boldsymbol{u} = r^2(\boldsymbol{u} \times \dot{\boldsymbol{u}}) \times \boldsymbol{u}$, or $\Theta\boldsymbol{v} = r^2\dot{\boldsymbol{u}}$, from which

$$\boldsymbol{X} = \dot{r}\boldsymbol{u} + (\Theta/r)\boldsymbol{v}. \tag{3}$$

Therefore, one may consider replacing the integration of $(\boldsymbol{x}, \boldsymbol{X})$ by that of $(r, \Theta, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{n})$. That is, the attitude of the orbital frame together with the in-plane motion.

Thus, from $\Theta\dot{\Theta} = \boldsymbol{G} \cdot \dot{\boldsymbol{G}}$, where, from Eq. (1), $\dot{\boldsymbol{G}} = \boldsymbol{x} \times \boldsymbol{P}$, we obtain

$$\dot{\Theta} = r\mathcal{T}, \tag{4}$$

with $\mathcal{T} \equiv \boldsymbol{P} \cdot \boldsymbol{v}$. On the other hand, straightforward operations yield

$$\dot{\boldsymbol{u}} = \frac{\Theta}{r^2}\boldsymbol{v}, \quad \dot{\boldsymbol{v}} = \frac{r}{\Theta}\mathcal{N}\boldsymbol{n} - \frac{\Theta}{r^2}\boldsymbol{u}, \quad \dot{\boldsymbol{n}} = -\frac{r}{\Theta}\mathcal{N}\boldsymbol{v}, \tag{5}$$

whereas, from differentiation of $\dot{r} = \dot{\boldsymbol{x}} \cdot \boldsymbol{u}$,

$$\ddot{r} = (-1 + p/r)(\mu/r^2) + \mathcal{R}, \tag{6}$$

with $\mathcal{R} \equiv \boldsymbol{P} \cdot \boldsymbol{u}$, thus completing the differential system.

Replacing the integration of Eq. (1) by that of Eqs. (4)–(6), with the six scalar constraints stemming from the orthonormality of the orbital frame, provides an important insight into the nature of perturbed Kepler motion. In particular, since $(\dot{\boldsymbol{u}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{n}}) = \boldsymbol{\omega} \times (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{n})$, we immediately obtain the angular velocity of the orbital frame

$$\boldsymbol{\omega} = (r/\Theta)\mathcal{N}\boldsymbol{u} + (\Theta/r^2)\boldsymbol{n}, \tag{7}$$

which shows that the orbital plane rotates slowly about the radial direction under the influence of the perturbation in the normal direction, whereas the orbital frame rotates fast in this plane with Keplerian-type motion. For the latter, the perturbation only acts indirectly through its effects on Θ and r, as follows from Eqs. (4) and (6).

On the other hand, the slow evolution of the attitude of the orbital plane is not suitably represented when using the fast rotating orbital frame. Therefore, it is more conveniently replaced by a slowly evolving frame, as the nodal frame $(O, \boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{n})$, where $\boldsymbol{\ell} = \boldsymbol{k} \times \boldsymbol{n}/\sin I$, where I denotes inclination, (the unit vector pointing to the ascending node), $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{\ell}$, or the apsidal frame $(O, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$, where $\boldsymbol{a} = \boldsymbol{e}/e$, where $e = \|\boldsymbol{e}\|$ denotes eccentricity, and $\boldsymbol{b} = \boldsymbol{n} \times \boldsymbol{a}$. From the definitions of the vectors defining these frames, straightforward operations yield their variations, from which we easily extract the corresponding angular velocities.

Thus, for the nodal frame $(\dot{\boldsymbol{\ell}}, \dot{\boldsymbol{m}}, \dot{\boldsymbol{n}}) = \boldsymbol{w} \times (\boldsymbol{\ell}, \boldsymbol{m}, \boldsymbol{n})$, and hence

$$\boldsymbol{w} = \frac{r}{\Theta}\mathcal{N}\boldsymbol{u} + \frac{r\sin\theta}{\Theta}\frac{\boldsymbol{k}\cdot\boldsymbol{n}}{\boldsymbol{k}\cdot\boldsymbol{m}}\mathcal{N}\boldsymbol{n}. \tag{8}$$

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As expected, the component in the radial direction is the same as before, yet now the one in the normal direction evolves slowly too, both driven by the perturbations in the out-of-plane direction. Regarding the apsidal frame $(\dot{\boldsymbol{a}}, \dot{\boldsymbol{b}}, \dot{\boldsymbol{n}}) = \tilde{\boldsymbol{\omega}} \times (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n})$, and hence

$$\tilde{\boldsymbol{\omega}} = \frac{r}{\Theta}\mathcal{N}\boldsymbol{u} + \frac{\Theta}{\mu}\frac{1}{e}\Big[2\mathcal{T}\sin f - \Big(\mathcal{R} + \frac{rR}{\Theta}\mathcal{T}\Big)\cos f\Big]\boldsymbol{n}, \quad (9)$$

with f denoting the true anomaly. The rotation in the out-of-plane direction is now much more involved, depending on the in-plane components of the perturbation, due to the dynamical definition of the apsidal frame as opposite to the geometric definitions of the orbital and nodal frames.

Therefore, we may ask ourselves, as Hansen did it first [7], why not to simplify things and choose a frame $(O, \boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{n})$ fixed in the orbital plane? That is, a rotating frame about the radial direction with angular velocity $\boldsymbol{\omega}^* = (r/\Theta)\mathcal{N}\boldsymbol{u}$. In that case the variation equations $(\dot{\boldsymbol{u}}^*, \dot{\boldsymbol{v}}^*, \dot{\boldsymbol{n}}) = \boldsymbol{\omega}^* \times (\boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{n})$, take the neat, symmetric form

$$\dot{\boldsymbol{u}}^* = -(\mathcal{N}/\Theta)(\boldsymbol{x}\cdot\boldsymbol{v}^*)\boldsymbol{n}, \quad \dot{\boldsymbol{v}}^* = (\mathcal{N}/\Theta)(\boldsymbol{x}\cdot\boldsymbol{u}^*)\boldsymbol{n}, \quad (10)$$

and $\dot{\boldsymbol{n}} = (\dot{\boldsymbol{u}}^* \cdot \boldsymbol{n})\boldsymbol{u}^* - (\dot{\boldsymbol{v}}^* \cdot \boldsymbol{n})\boldsymbol{v}^*$. The initial orientation $\boldsymbol{u}^* = \boldsymbol{u}^*(0)$, $\boldsymbol{v}^* = \boldsymbol{v}^*(0)$, can be chosen arbitrarily. The instantaneous direction $\boldsymbol{u}^*$ is customarily denoted as a *departure point* in the orbital frame [8, 9].

Differentiation of $\cos\vartheta = \boldsymbol{u} \cdot \boldsymbol{u}^*$, followed by the replacement of $\dot{\boldsymbol{u}}$ and $\dot{\boldsymbol{u}}^*$ from previous equations, yields the rotation $\dot{\vartheta} = \Theta/r^2$, which, as expected, matches the normal component of the angular velocity of the orbital frame in Eq. (7). This can be viewed as a Keplerian rotation as far as the perturbation has only indirect effects on $\dot{\vartheta}$.

Remarkably, in this slowly rotating frame $\boldsymbol{\omega}^* \times \boldsymbol{x} = \boldsymbol{0}$, and hence the theorem of the moving frame trivially shows that the velocity is the same in both the inertial and ideal frames. That is, calling $x^* = \boldsymbol{x} \cdot \boldsymbol{u}^* = r\cos\vartheta$, $y^* = \boldsymbol{x} \cdot \boldsymbol{v}^* = r\sin\vartheta$, in the ideal frame the first of Eq. (1) yields $\dot{x}^* = \dot{\boldsymbol{x}} \cdot \boldsymbol{u}^* = \boldsymbol{X} \cdot \boldsymbol{u}^* = X^*$, $\dot{y}^* = \dot{\boldsymbol{x}} \cdot \boldsymbol{v}^* = \boldsymbol{X} \cdot \boldsymbol{v}^* = Y^*$. Moreover, from Eq. (3), $\boldsymbol{\omega}^* \times \boldsymbol{X} = \mathcal{N}\boldsymbol{n}$. Then, the theorem of the moving frame yields $(\dot{\boldsymbol{X}})_{\boldsymbol{\omega}^*} = \dot{\boldsymbol{X}} - \mathcal{N}\boldsymbol{n}$. Therefore, the in-plane motion with respect to the ideal frame is directly obtained from the second of Eq. (1). By these reasons, Hansen named the intermediate frame $(O, \boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{n})$ *ideal* [7].

On account of $\boldsymbol{v} \cdot \boldsymbol{u}^* = -\boldsymbol{u} \cdot \boldsymbol{v}^*$, $\boldsymbol{v} \cdot \boldsymbol{v}^* = \boldsymbol{u} \cdot \boldsymbol{u}^*$, we readily obtain

$$\dot{X}^* = \dot{\boldsymbol{X}} \cdot \boldsymbol{u}^* = -(\mu/r^3)x^* + (\mathcal{R}x^* - \mathcal{T}y^*)/r,$$
$$\dot{Y}^* = \dot{\boldsymbol{X}} \cdot \boldsymbol{v}^* = -(\mu/r^3)y^* + (\mathcal{R}y^* + \mathcal{T}x^*)/r.$$

It may be checked that these variations agree with Eqs. (11_1)–(11_4) in [6], whose derivation was based on exterior calculus. The additional term in [6] is recovered when making the decomposition $\boldsymbol{P} = \boldsymbol{F} + \nabla_{\boldsymbol{x}} U$. Note that $\Theta = x^*Y^* - X^*y^*$.

In practice, the over-dimensioned integration of the attitude variations $(\dot{\boldsymbol{u}}^*, \dot{\boldsymbol{v}}^*, \dot{\boldsymbol{n}})$ is replaced by that of the Euler angles (Ω, I, β), standing for right ascension of the ascending node, inclination, and polar angle $\beta = \arccos(\boldsymbol{u}^* \cdot \boldsymbol{\ell})$, respectively. The usual rotation formula $\boldsymbol{\omega}^* = \dot{\Omega}\boldsymbol{k} + \dot{I}\boldsymbol{\ell} + \dot{\theta}\boldsymbol{n}$ [10, 11], allows us to obtain the components $(\boldsymbol{\omega}^* \cdot \boldsymbol{u}, \boldsymbol{\omega}^* \cdot \boldsymbol{v}, \boldsymbol{\omega}^* \cdot \boldsymbol{n})$, from which

$$\dot{\Omega} = \frac{r}{\Theta}\mathcal{N}\frac{\sin\theta}{\sin I}, \quad \dot{I} = \frac{r}{\Theta}\mathcal{N}\cos\theta, \quad \dot{\beta} = -\dot{\Omega}\cos I$$

where $\theta = \vartheta + \beta$. Moreover, to avoid singularities, it is customary after Musen [12] to replace the integration of the Euler angles by that of the Euler parameters. Finally, after Deprit [2], the integration of Cartesian coordinates in the ideal frame is very effectively replaced by the integration of the three hodographic velocities $\kappa = (\mu/\Theta)\boldsymbol{e} \cdot \boldsymbol{u}^*$, $\sigma = (\mu/\Theta)\boldsymbol{e} \cdot \boldsymbol{v}^*$, $\zeta = \mu/\Theta$ [13].

Using internal units, modulating ϑ between 0 and 2π, and keeping the geometric property stemming from the norm 1 of the quaternion comprised by the Euler parameters are common programming strategies that increase the performance of the numerical integration.

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