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The minimal tautologies play main role in the proof complexity area. In fact all propositional formulaes, proof complexities of which are investigated in many well known papers, are minimal tautologies. There is a traditional assumption that minimal tautology shouldn't be more complicated than any substitution in it, that is must be some monotonicity of proofs. This idea was first raised by Anikeev in [1]. He has given two types of not complete propositional proof systems, in the first of which the proof lines of all minimal tautologies are not more, than the proof lines for every results of substitutions in them, and the second one, in which the proof lines of substituted formulas can be less than the proof lines of some corresponding minimal tautologies.

At first we showed that for classical, intuitionistic, Johansson's, monotone two-valued logics and for different many-valued logics the number of minimal tautologies for a given tautology of size n can be exponential function in n. Then for some propositional proof systems of mentioned logics we investigate the relations between the lines (*t-**complexities***) and sizes (*l-**complexities***) of proofs for minimal tautologies of this logic and for results of a substitutions in them. We introduced the notions of *monotonous* and *strongly monotonous* properties for the proof systems and investigated these properties for many well known propositional proof systems of different two-valued and many-valued logics, as well as for some new systems, constructed for mentioned logics by us.

Definition 1. A tautolgy is called ***minimal in this logic*** if replacement result of all occurrences for each its non-elementary subformulas by some new variable is not a tautology of this logic.

Let $A_n = \overbrace{p \wedge (p \wedge (p \wedge \ldots \wedge (p \wedge p) \ldots))}^{n}$. For tautologies $B_n = p \supset (p \vee p) \vee A_n$, the following tautologies $C_n = p \supset q \vee A_n$ and $D_n = p \supset (p \vee p) \vee r$ are minimal (for monotone logic the sequents $B_n = p \to (p \vee p) \vee A_n$, $C_n = p \to q \vee A_n$ and $D_n = p \to (p \vee p) \vee r$ accordingly). It is not difficult to see, that tautologies C_n are "harder" than D_n and B_n.

For every minimal tautology φ of fixed logic, by $S(\varphi)$ is denoted the set of all tautologies, which are results of a substitution in φ.

For any proof system ϕ and tautology φ we denote by $t^{\phi}(\varphi)$ ($l^{\phi}(\varphi)$) the minimal possible value of lines (sizes) for all ϕ-proofs of tautology φ.

Definition 2. The proof system ϕ of some logic is called *t-**monotonous*** $/t-m/$ (*l-**monotonous*** $/l-m/$), if for every non-minimal tautology ψ of this logic there is a minimal tautology φ of the same logic such that $\psi \in S(\varphi)$ and $t^{\phi}(\psi) = t^{\phi}(\varphi)$ ($l^{\phi}(\psi) = l^{\phi}(\varphi)$).

Definition 3. The proof system ϕ of some logic is called *t-**strongly monotonous*** $/t-sm/$ (*l-**strongly monotonous*** $/l-sm/$), if for every tautology ψ of this logic there is no minimal tautology φ of the same logic such that, $\psi \in S(\varphi)$ and $t^{\phi}(\varphi) > t^{\phi}(\psi)$ ($l^{\phi}(\varphi) > l^{\phi}(\psi)$).

We investigated the above properties for different well-known propositional proof systems of classical two-valued logic (CL), intuitionistic, Johansson's, monotone two-valued logics (IL, JL, MonL) for different many-valued logics (MVL) and for some new systems, constructed for mentioned logics by us (the definitions of these systems are in Appendix).

Our main results:

1. The systems, based on generalization of splitting method for CL and MVL, as well as eliminations systems, based on the determinative normal forms for CL, IL, JL and MVL are neither $t-m$ $(l-m)$ and therefore not $t-sm$ $(l-sms)$ [2, 3].

2. The resolution systems for CL, IL, JL, and cut-free sequent systems for CL, IL, JL and MonL are $t-m$ $(l-m)$, but not $t-sm$ $(l-sms)$ [4, 5].

3. The sequent systems with cut rule and Frege systems for CL, IL, JL are neither $t-m$ and therefore not $t-sm$: it is showed that for each logic there is a sequence of tautologies ψ_n, every of which has **unique** minimal tautology φ_n such, that for every n the proofs lines of φ_n in pointed systems are by order more than the the proofs lines of ψ_n in these systems [6, 7, 8, 9].

4. All complete well known systems of the above mentioned logics are not $t-sm$ $(l-sms)$.

The question about existence of some strongly monotonous system are still open.

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Appendix

Here we give the definitions of two systems, mentioned above. Following the usual terminology we call the variables and negated variables *literals*. The conjunct K (term) can be represented simply as a set of literals (no conjunct contains a variable and its negation simultaneously). Let φ be a propositional formula, $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all variables of φ, and $P' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\}$ $(1 \le m \le n)$ be some subset of P.

Definition 1.1. Given $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\} \subset E^m$, the conjunct $K^\sigma = \{p_{i_1}^{\sigma_1}, p_{i_2}^{\sigma_2}, \ldots, p_{i_m}^{\sigma_m}\}$ is called $\varphi-$ 1-determinative ($\varphi-$ 0-determinative) if assigning σ_j $(1 \le j \le m)$ to each p_{ij} we obtain the value of φ (1 or 0) independently of the values of the remaining variables.

Definition 1.2. DNF $D = \{K_1, K_2, \ldots, K_j\}$ is called determinative DNF (DDNF) for φ if $\varphi = D$ and every conjunct K_i $(1 \le i \le j)$ is 1-determinative for φ.

Elimination system for PC (EC)

The axioms of **EC** aren't fixed, but for every formula φ each conjunct from some dDNF of φ can be considered as an axiom.The *elimination rule* (e-rule) infers $K' \cup K''$ from conjuncts $K' \cup \{p\}$ and $K' \cup \{\bar{p}\}$, where K' and K'' are conjuncts and p is a variable. The proof in **EC** is a finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of EC or is inferred from earlier conjuncts in the sequence by e-rule. DNF $D = \{K_1, K_2, \ldots, K_l\}$ is tautology if using e-rule the empty conjunction $(\emptyset)$ can be proved from the axioms $\{K_1, K_2, \ldots, K_l\}$. It is obvious that the system **EC** is complete.

Generalised Splitting system GS.

Let φ be some formula and p be some of its variable. Results of splitting method of formula φ by variable p (splinted variable) are the formulas $\varphi[p^\delta]$ for every δ from the set $\{0, 1\}$, which are obtained from φ by assigning δ to each occurrence of p and successively using the elementary equivalences of logical functions. Note that, in some cases, the formulas $\varphi[p^\delta]$ can remain after pointed transformation occurrences of the constant δ as well. The generalization of splitting method allow as associate with every formula φ some tree with root, nodes of which are labeled by formulas and edges, labeled by literals. The root is labeled by itself formula φ. If some node is labeled by formula v and α is some its variable, then both edges, which going out from this node, are labeled by one of literals α^δ for every δ from the set $\{0, 1\}$, and every of 2 "sons" of this node is labeled by corresponding formula $v[\alpha^\delta]$. Each of the trees leafs is labeled with some constant from the set $\{0, 1\}$. The tree, which is constructed for formula φ by described method, we will call *splitting tree* of φ in future.

The **GS** proof system can be defined as follows: for every formula φ must be constructed some splitting tree and if all tree's leafs are labeled by the value 1, then formula φ is tautology and therefore we can consider the pointed constant 1 as an axiom, and for every formula v, which is label of some splitting tree node, and p is its splinted variable, then as some inference rule can be consider the following figure $v[p^0]$, $v[p^1] \vdash v$, therefore every above described splitting tree can be consider as some proof of φ in the system **GS**.