Finite Hilbert-Style Axiomatizations of Disjunctive and Implicative Finitely-Valued Logics with Equality Determinant

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Abstract. Here, we, first of all, develop a universal method of [effective] constructing a [finite] Hilbert-style axiomatization of the logic of a given finite disjunctive/implicative matrix with equality determinant [and finitely many connectives]. In addition, using same auxiliary tools, we prove that the lattice of disjunctive extensions of the logic of a finite [more specifically, one-element] class of finite disjunctive matrices [with equality determinant] is dual to the distributive lattice of all ({strict} Horn) universal relative [i.e., relatively hereditary] subclasses of the class of all *consistent* — viz., having non-distinguished values — submatrices of defining matrices, ⟨finite⟩ relative axiomatizations of the latter ones ⟨[to be found effectively]⟩ being analytically transformed to those of the former ones.

1. Introduction

Though various universal approaches to (mainly, *many-place*) *sequent* axiomatizations of finitely-valued logics (cf., *e.g.*, [21] for a most universal approach subsuming all preceding ones, in their turn, going back to the independent works [27] and [26] originating this area of Proof Theory for Many-Valued Logic) have being extensively developed, the problem of their standard (viz., Hilbert-style) axiomatizations (especially, on a generic level) has deserved much less emphasis despite of the problem's being especially acute within both General Logic and Proof Theory.

On the other hand, the general study [19], equally subsumed by [21], has suggested a universal method of [effective] constructing a multi-conclusion Gentzen-style (viz., *two-side* sequent) axiomatization with structural rules and Cut Elimination Property of the logic of a given finite matrix with equality determinant — viz., a set of secondary unary connectives discriminating distinct truth values of the matrix by the values of one the former ones on the latter ones' being distinguished — [and finitely many connectives] (in particular, any four-valued expansion of Dunn-Belnap's "useful" four-valued logic [2, 3] [by finitely many connectives as well as Łukasiewicz finitely-valued logics [9, 11]]). In this work, providing the matrix involved is disjunctive/implicative (that equally covers aribitrary/implicative four-valued expansions of Dunn-Belnap's four-valued logic/" as well as Łukasiewicz finitely-valued logics), we enhance the mentioned study by [effective] transforming any [finite] *sequential table for* the matrix (viz., a collection of context-free skeletons of uniquely-chosen introduction rules for the matrix and all *compound* non-constant connectives — viz., values of elements of the equality determinant on primary non-constant connectives — not belonging to the equality determinant) and minimal — under the subsuming quasi-ordering, while treating sequents as first-order clauses (cf. [25]) — sequent axioms with disjoint sides consisting of solely either elements of the equality determinant or their values on constant connectives true in the matrix, actually giving a Gentzen-style axiomatization of the logic of the matrix in [19], to a [finite] Hilbert-style axiomatization of the logic.

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It appears that practically same auxiliary tools, concerning sequent calculi, going back to [17], advanced here and used for solving the problem described above, are equally applicable to that of finding disjunctive extensions of disjunctive not necessarily uniform/unitary finitely-valued logics not necessarily with equality determinant as well as their finite both matrix semantics and relative axiomatizations, so we solve this problem as well, laying a special emphasis onto the unitary case with equality determinant providing the effectiveness of the proposed solution.

The rest of the material is as follows. Its exposition is entirely self-contained. Section 2 is a concise summary of basic issues underlying the work. In Section 3, upon the basis of the rather conventional paradigm "rules as purely-single-conclusion two-side sequents", under which logics (formally as finitary rather Tarski-style consequence relations than, equivalently, closure operators) are nothing but calculi closed under purely-single-conclusion two-side sequent structural rules — Reflexivity, Cut and Subsuming[1] (in its turn, subsuming the traditional one — Enlargement; Permutation and Contraction being implicit, due to treating sequent sides as finite rather sets than sequences), we propose a really elegant formalism uniformly covering both Hilbert- and Gentzen-style propositional calculi (in particular, axiomatizing propositional logics) as well as providing a quite transparent insight into the issue of their matrix semantics going back to [8] and [30], respectively. Then, in Section 4 we develop/recall certain advanced key issues concerning disjunctivity/implicativity used here. Next, Sections 5 and 6 are entirely devoted to the main general results of the work (cf. the Abstract) further exemplified in Section 7. Finally, Section 8 summarizes principal contributions of the work.

2. General mathematical background

2.1. **Set-theoretical background.** As usual, natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser ones), the ordinal of all them being denoted by ω. Then, given any $N \subseteq \omega$ and any $n \in (\omega \setminus 1)$, set $(N \div n) \triangleq \{\frac{i}{n} \mid i \in N\}$. Likewise, functions are treated as binary relations. Finally, any singleton is identified with its unique element, unless any confusion is possible.

Let S, T and U be sets. Then, an enumeration of S is any bijection from its cardinality $|S|$ onto S. Next, the set of all subsets of S (including T) [of cardinality in $\alpha \subseteq \omega$] is denoted by $\wp_{[\alpha]}((T,)S)$, respectively. Further, in case $T \subseteq S^S$ and $U \subseteq S$, put $T[U] \triangleq \{f(a) \mid f \in T, a \in U\}$. As usual, any *$S$-tuple* (viz., a function with domain S) is normally written in the sequence form $\bar{t}$, its *s-th component* (viz., value under argument s), where $s \in S$, being written as either t_s or, to avoid double subscripts, t^s. Put $\eth_S \triangleq \{\langle s, s\rangle \mid s \in S\}$, relations of such a kind being said to be *diagonal*, and $S^{*|+} \triangleq (\bigcup_{i \in (\omega \setminus (0|1))} S^i)$, elements of which being treated as |non-empty finite tuples constituted by elements of S. Then, any $\diamond : S^2 \to S$ determines the equally-denoted mapping $\diamond : S^+ \to S$ defined by induction on the length of elements of S^+ as follows: for any $b \in S$ [and any $\bar{a} \in S^+$], set $(\diamond\langle[\bar{a},]b\rangle) \triangleq ([(\diamond\bar{a})\diamond]b)$. In particular, given any $f \in S^S$ and any $n \in \omega$, $f^n \triangleq (\circ\langle n \times \{f\}, \eth_S\rangle) \in S^S$ is called the *n-th degree of f*. Likewise, f determines the equally denoted mapping $f : S^* \to S^*, \bar{a} \mapsto (f \circ \bar{a})$. Then, f is said to be *R-[anti-]monotonic*, provided $f[R \cap S^2] \subseteq R^{[-1]}$. Furthermore, given any $\diamond : (S \times T) \to S$ and any $b \in T$, set $(\diamond b) : S \to S, a \mapsto (a \diamond b)$. Finally, any $\diamond : (S \times T) \to T$ determines the equally-denoted mapping $\diamond : (S^* \times T) \to T$ by induction on the length of elements of S^* as follows: for any $b \in T$ [and any $a \in S$ {as well as any $\bar{c} \in S^*$}], set $(\langle[\{\bar{c},\}a]\rangle \diamond b) \triangleq [\{(\bar{c}\diamond\}(a\diamond]b[)\{)\}]$. In general, any $B \subseteq \wp(S)$ is identified with the poset $\langle B, \subseteq \cap B^2\rangle$. Then, an *anti-chain of B* is any $A \subseteq B$ such that, for all

[1]Cf. [25] for roots of this term.

$X, Y \in A$, it holds that $(X \subseteq Y) \Rightarrow (X = Y)$. Likewise, a *lower cone of* B is any $C \subseteq B$ such that, for all $X \in C$, it holds that $(B \cap \wp(X)) \subseteq C$. Clearly, providing B is finite (in particular, S is so), $C \mapsto \max(C)$ and $A \mapsto (B \cap \bigcup\{\wp(X) \mid X \in A\})$ are inverse to one another bijections between the sets of all lower cones and of all anti-chains of B.

2.2. **Algebraic background.** In general, to unify notations, unless otherwise specified, abstract algebras are denoted by capital Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding capital Italic letters [with same indices, if any].

Let Σ be an algebraic (viz., functional) signature, constituted by operation symbols of finite arity treated as *(propositional/sentential)* $\{$*primary*$\}$ *connectives*, the set of all n-ary ones, where $n \in \omega$, being denoted by $\Sigma{\restriction}n$. Likewise, given any $\alpha \in (\{\omega\}[\cup\omega])$, elements of the set $\mathrm{Var}_{\omega[\cap\alpha]} \triangleq (\mathrm{img}\,\bar{x}_{\omega[\cap\alpha]})$, where $\bar{x}_{\omega[\cap\alpha]} \triangleq \langle x_i\rangle_{i\in(\omega[\cap\alpha])}$, are viewed as *(propositional/sentential) variables [of rank* α*]*. Then, [in case $\alpha \neq 0$, whenever $(\Sigma{\restriction}0) = \varnothing$] we have the absolutely-free Σ-algebra $\mathfrak{Tm}^{[\alpha]}_\Sigma$ freely-generated by the set $\mathrm{Var}_{\omega[\cap\alpha]}$ with carrier denoted by $\mathrm{Tm}^{[\alpha]}_\Sigma \supseteq \mathrm{Var}_{\omega[\cap\alpha]}$, whose elements are called Σ*-terms [of rank* α*]* and are viewed as *(propositional/sentential)* Σ*-formulas [of rank* α*]*. Next, the function Var with domain Tm_Σ assigning the finite set of all variables *actually* occurring in an argument Σ-term φ is defined by induction on construction of φ with diagonal (under the identification of singletons with their unique elements) restriction on Var_ω and setting $\mathrm{Var}(F(\bar{\varphi}) \triangleq (\bigcup \mathrm{Var}[\mathrm{img}\,\bar{\varphi}])$, for all $F \in \Sigma$ of arity $n \in \omega$ and all $\bar{\varphi} \in (\mathrm{Tm}_\Sigma)^n$. Further, a *secondary* n*-ary connective of* Σ, where $n \in \omega$, is any Σ-term of rank $n+(1-\min(1, \max(n, |\Sigma_0|)))$, any primary n-ary connective F of Σ being identified with the secondary one $F(\bar{x}_n)$, for the sake of unification. Furthermore, given any $T \subseteq \mathrm{Tm}_\Sigma$ [and any non-empty $\alpha \subseteq \omega$], the set $\mathrm{Tm}^{[\alpha]}_T \subseteq \mathrm{Tm}^{[\alpha]}_\Sigma$ of T*-terms [of rank* α*]* is defined in the standard recursive manner by means of variables [of rank α] and Σ-terms (viz., secondary connectives of Σ) in T. (More precisely, $\mathrm{Tm}^{[\alpha]}_T \triangleq (\bigcap\{S \in \wp(\mathrm{Var}_{\omega[\cap\alpha]}, \mathrm{Tm}^{[\alpha]}_\Sigma) \mid \forall\sigma \in \hom(\mathfrak{Tm}_\Sigma, \mathfrak{Tm}^{[\alpha]}_\Sigma) : (\sigma[\mathrm{Var}_\omega] \subseteq S) \Rightarrow (\sigma[T] \subseteq S)\}) \in \wp(T, \mathrm{Tm}^{[\alpha]}_\Sigma)$.) Finally, any homomorphism h from $\mathrm{Tm}^\alpha_\Sigma$ [to itself], being uniquely determined by $h' \triangleq (h{\restriction}(\mathrm{Var}_\alpha[\setminus V]))$ [where $V \subseteq \mathrm{Var}_\alpha$ such that $h{\restriction}V$ is diagonal], is identified with h', in its turn, often written in the conventional assignment [resp., substitution] form $[v/h'(v)]_{v\in(\mathrm{dom}\,h')}$.

2.2.1. *Logical matrices.* As usual, any *(logical)* Σ*-matrix* (cf., e.g., [8]), i.e., a couple of the form $\mathcal{A} = \langle\mathfrak{A}, D^{\mathcal{A}}\rangle$ with its *underlying [*Σ*-]algebra* $\mathfrak{A}$ and its *truth predicate* (viz., the set of its *distinguished values*) $D^{\mathcal{A}} \subseteq A$, is treated as a first-order model structure (viz., an algebraic system; cf. [10]) of the signature $\Sigma \cup \{D\}$ with single unary predicate D, in which case the notion of a *submatrix of* $\mathcal{A}$ (in particular, the one of the *restriction* $(\mathcal{A}{\restriction}B)$ *of* $\mathcal{A}$ *on* any $B \subseteq A$ forming a subalgebra of $\mathfrak{A}$ as the submatrix of $\mathcal{A}$ with underlying algebra $\mathfrak{A}{\restriction}B$) is defined in the standard way as any Σ-matrix of the form $\langle\mathfrak{B}, D^{\mathcal{A}} \cap B\rangle$, where $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, while, for any $\Sigma' \subseteq \Sigma$, $(\mathcal{A}{\restriction}\Sigma') \triangleq \langle\mathfrak{A}{\restriction}\Sigma', D^{\mathcal{A}}\rangle$. (In general, to unify notations, unless otherwise specified, logical matrices are denoted by capital Calligraphic letters [possibly, with indicies], their underlying algebras being denoted by corresponding capital Fraktur letters [with same indicies, if any].) This is said to be *consistent*, whenever $D^{\mathcal{A}} \neq A$. Likewise, it is said to be $\diamond$*-disjunctive/-implicative*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided, for all $a, b \in A$, it holds that $((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}}) \Leftrightarrow ((a \notin / \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}}))/$", in which case it is $\veebar_\diamond$-disjunctive, where $(x_0 \veebar_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$", and so is any submatrix of $\mathcal{A}$. Finally, according to [19], an *equality determinant for* $\mathcal{A}$ is any $\Im \subseteq \mathrm{Tm}^1_\Sigma$ such that every $a, b \in A$ are equal, whenever, for each $\iota \in \Im$, it holds that $(\iota^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\iota^{\mathfrak{A}}(b) \in D^{\mathcal{A}})$, in which case it is so for any submatrix of $\mathcal{A}$, and so is any/"some finite" $\Im' \subseteq \mathrm{Tm}_\Sigma^1$ such that $\Im' \supseteq / \subseteq \Im$/", in case A is finite". Then, given any $a \in A$, set $\Im^{\mathcal{A}}_{a,+|-} \triangleq \{\iota \in \Im \mid \iota^{\mathfrak{A}}(a) \in | \notin D^{\mathcal{A}_n}\}$, respectively.

3. Abstract languages and their sequentializations

A(n) *(abstract) language* is any couple of the form $L \triangleq \langle \mathrm{Fm}_L, \mathrm{Sb}_L \rangle$, where Fm_L is a set, whose elements are called *L-formulas*, and $\mathrm{Sb}_L \subseteq \mathrm{Fm}_L^{\mathrm{Fm}_L}$ contains $\eth_{\mathrm{Fm}_L}$ and is closed under composition $\circ$, whose elements are called *L-substitutions*. Then, an *L-substitutional instance of* a $\Phi \in \mathrm{Fm}_L$ is any L-formula of the form $\sigma(\Phi)$, where $\sigma \in \mathrm{Sb}_L$. Next, [given any $\alpha \subseteq \omega$] any $\langle \Gamma, \Delta \rangle \in \mathrm{Seq}_L^{[\alpha]} \triangleq (\wp_\omega(\mathrm{Fm}_L) \times \wp_{\omega[\cap\alpha]}(\mathrm{Fm}_L))$ is called an *[α-conclusion] L-sequent* and normally written as $\Gamma \vdash \Delta$, elements of Γ/Δ being referred to as *premises/conclusions of* it, "(purely-)multi|single" standing for "$(\omega|2)(\setminus 1)$", respectively, $\vdash \Gamma$ and $\Delta \vdash$ standing for $\varnothing \vdash \Gamma$ and $\Delta \vdash \varnothing$, respectively, as usual. This is said to be *disjoint*, whenever Γ and Δ are so. Further, an *L-rule/-axiom* is any purely-single-conclusion L-sequent $\Gamma \vdash \Phi$ /"without premises" in which case it is often written in the displayed form $\frac{\Gamma}{\Phi}$ /"and identified with Φ", sets of them being referred to as */axiomatic L-calculi*. Then, given any $\Phi, \Psi \in \mathrm{Fm}_L$, $\frac{\Phi}{\Psi}\updownarrow$ stands for $(\Phi \dashv\vdash \Psi) \triangleq \{\Phi \vdash \Psi; \Psi \vdash \Phi\}$. Furthermore, any $X \subseteq \mathrm{Fm}_L$ is said to be *closed under* an L-sequent $\Gamma \vdash \Delta$, provided $(\Gamma \subseteq X) \Rightarrow ((\Delta \cap X) \neq \varnothing)$. Finally, any unary operation f on Fm_L (including L-substitutions) determines the equally-denoted mapping $f : \mathrm{Seq}_L^{[\alpha]} \to \mathrm{Seq}_L^{[\alpha]}, (\Gamma \vdash \Delta) \mapsto (f[\Gamma] \vdash f[\Delta])$. In this way, $\$_{[\alpha]}(L) \triangleq \langle \mathrm{Seq}_L^{[\alpha]}, \mathrm{Sb}_L \rangle$ is a language, called the *[α-conclusion] sequentialization of L*, "[α-conclusion] (n-order) sequent|Gentzen-style L-" standing for "$\$_{[\alpha]}^{(n)}(L)$-" (where $n \in (\omega \setminus 2)$). Then, an L-rule $\mathcal{R} = (\Gamma \vdash \Phi)$ is said to be *derivable in* an L-calculus $\mathcal{C}$, provided there is a *$\mathcal{C}$-derivation of* $\mathcal{R}$, that is, a proof $\bar{d} \in \mathrm{Fm}_L^*$ of Φ by means of axioms in Γ (to be treated as *hypotheses*) and L-rules in $\mathrm{Sb}_L[\mathcal{C}]$, in which case, for any $\sigma \in \mathrm{Sb}_L$, $\sigma \circ \bar{d}$ is a $\mathcal{C}$-derivation of $\sigma(\mathcal{R})$, because Sb_L is closed under composition, while $\langle \Gamma, \Phi \rangle$ is a $\mathcal{C}$-derivation of $\mathcal{R}$, for $\eth_{\mathrm{Fm}_L} \in \mathrm{Sb}_L$. Likewise, it is said to be *admissible in* $\mathcal{C}$, provided any L-axiom is derivable in $\mathcal{C}$, whenever this is derivable in $\mathcal{C} \cup \{R\}$, that is, the set of all L-axioms derivable in $\mathcal{C}$ is closed under every L-substitutional instance of $\mathcal{R}$.

We use the following "sign sequent" notation: given any $i \in 2$ and any $\Gamma \in \wp_\omega(\mathrm{Fm}_L)$, put $(i : \Gamma) \triangleq \{\langle i, \Gamma \rangle, \langle 1 - i, \varnothing \rangle\} \in \mathrm{Seq}_L$.

Given two L-sequents $\Phi = (\Gamma \vdash \Delta)$ and $\Psi = (\Lambda \vdash \Theta)$, we have their *sequent disjunction* $(\Phi \uplus \Psi) \triangleq (((\Gamma \cup \Lambda) \vdash (\Delta \cup \Theta)) \in \mathrm{Seq}_L$. Likewise, we have their *sequent implication* $(\Phi \rhd \Psi) \triangleq \{\Psi \uplus (0 : \psi) \mid \psi \in \Delta\} \cup \{\Psi \uplus (1 : \phi) \mid \phi \in \Gamma\}) \in \wp_\omega(\mathrm{Seq}_L)$, in which case, for any $\Omega \in \wp_\omega(\mathrm{Seq}_L)$ we set $(\Phi \rhd \Omega) \triangleq (\bigcup(\lhd\Phi)[\Omega])$. Finally, Φ is said to *[diagonally] subsume* Ψ ($\Phi \preceq_{[\eth]} \Psi$, in symbols), provided there is some $\sigma[= \eth_{\mathrm{Fm}_L}] \in \mathrm{Sb}_L$ such that both $\sigma[\Gamma] \subseteq \Lambda$ and $\sigma[\Delta] \subseteq \Theta$, in which case $\preceq_{[\eth]}$ is a quasi-ordering [more specifically, partial ordering] on Seq_L.

Then, a sequent L-calculus $\mathcal{G}$ is said to be *[deductively] multiplicative*, provided, for every sequent L-rule $\mathcal{R}$ [derivable] in $\mathcal{G}$ and each L-sequent Ψ, $(\uplus\Psi)(\mathcal{R})$ is derivable in $\mathcal{G}$.

The following sequent L-rules are said to be *[native] structural*:

Reflexivity	$\Phi \vdash \Phi$
[Diagonal] Subsuming	$\frac{\Phi}{\Psi} \quad (\Phi \preceq_{[\eth]} \Psi)$
Cut	$\dfrac{\{(\Lambda \cup \Gamma) \vdash (\Delta \cup \{\Phi\}), (\Gamma \cup \{\Phi\}) \vdash (\Delta \cup \Theta)\}}{(\Lambda \cup \Gamma) \vdash (\Delta \cup \Theta)}$

where $\Lambda, \Gamma, \Delta, \Theta \in \wp_\omega(\mathrm{Fm}_L)$ and $\Phi, \Psi \in \mathrm{Fm}_L$, the set of all ($\alpha$-conclusion of) them (where $\alpha \subseteq \omega$) being denoted by $[\mathcal{N}]\mathcal{S}_L^{(\alpha)}$, respectively. {Instances of Diagonal Subsuming with distinct premise and conclusion are nothing but instances of *multiple* Enlargement.} Likewise, the set of all instances of Diagonal Subsuming and Reflexivity/Cut is denoted by $(\mathcal{R}/\mathcal{C})\mathcal{DS}_L$, respectively.

Lemma 3.1 (Sequent Deduction Theorem; cf. Theorem 4.2 of [17]). *Let $\mathcal{G}$ be a sequent L-calculus, $\Omega \in \wp_\omega(\mathrm{Seq}_L)$ and $\Phi, \Psi \in \mathrm{Seq}_L$. Suppose Diagonal Subsuming as well as Cut/Reflexivity are derivable in $\mathcal{G}$ (while this is deductively multiplicative). Then, $\frac{\Omega \cup \{\Phi\}}{\Psi}$ is derivable in $\mathcal{G}$ if/(only if), for each $\Upsilon \in (\Phi \rhd \Psi)$, $\frac{\Omega}{\Upsilon}$ is so.*

Proof. Let $\Phi = (\Gamma \vdash \Delta)$. Consider any $[(\Lambda|\Theta) \subseteq](\Gamma|\Delta) \ni \varphi[\notin (\Lambda|\Theta)]$. [Then, both $\frac{\Phi}{\Psi \uplus \Phi}$ and $\frac{\{\Psi \uplus ((1|0) : \varphi); (\Lambda \vdash \Theta) \uplus ((0|1) : \varphi)\}}{\Psi \uplus (\Lambda \vdash \Theta)}$ are derivable in $\mathcal{CDS}_L$. In this way, the "if" part is by induction on $|\Gamma \setminus \Lambda| + |\Delta \setminus \Theta|$, for $\Psi = (\Psi \uplus (\vdash))$.] Conversely, $\Phi \uplus ((1|0) : \varphi)$ is derivable in $\mathcal{RDS}_L$. Therefore, once $\mathcal{G}$ is deductively multiplicative, by Diagonal Subsuming, $\frac{\Omega}{\Psi \uplus ((1|0) : \varphi)}$ is derivable in $\mathcal{G}$, whenever $\frac{\Omega \cup \{\Phi\}}{\Psi}$ is so, as required. $\square$

An *L-logic* is any L-calculus closed under each element of $\mathcal{S}_L^{2\setminus 1}$. This is said to be *[in]consistent*, whenever it is [not] distinct from $\mathrm{Seq}_L^{2\setminus 1}$. Given any [sequent] L-calculus $\mathcal{C}$ [in which each element of $\mathcal{NS}_L^{2\setminus 1}$ is admissible], the set $\mathcal{L}_\mathcal{C}$ of all those L-rules, which are derivable in $\mathcal{C}$, is an L-logic said to be *axiomatized by* $\mathcal{C}$. (Clearly, any L-logic is axiomatized by itself.) Then, a *[proper] extension of* an L-logic $\mathcal{L}$ is any L-logic $\mathcal{L}' \supseteq \mathcal{L}$ [distinct from] $\mathcal{L}$, in which case $\mathcal{L}$ is refereed to as a *[proper] sublogic of* $\mathcal{L}'$. This is said to be *axiomatized by* an L-calculus $\mathcal{C}'$ *relatively to* $\mathcal{L}$, whenever it is axiomatized by $\mathcal{L} \cup \mathcal{C}'$, that is, by $\mathcal{C} \cup \mathcal{C}'$, where $\mathcal{C}$ is any L-calculus axiomatizing $\mathcal{L}$. An extension $\mathcal{L}$‘ of $\mathcal{L}$ is said to be *axiomatic*, whenever it is relatively axiomatized by an axiomatic L-calculus $\mathcal{A}$, that is, by the set of all L-axioms in $\mathcal{L}'$, in which case:

$$\mathcal{L}' = \{\Phi \in \mathrm{Seq}_L^{2\setminus 1} \mid \exists \Gamma \in \wp_\omega(\mathrm{Sb}_L[\mathcal{A}]) : ((0 : \Gamma) \uplus \Phi) \in \mathcal{L}\}. \tag{3.1}$$

3.1. **Sentential languages, calculi and logics.** Let Σ be an algebraic signature. Then, $L_\Sigma \triangleq \langle \mathrm{Tm}_\Sigma, \hom(\mathfrak{Tm}_\Sigma, \mathfrak{Tm}_\Sigma)\rangle$ is an abstract language, called the *(propositional/sentential/Hilbert-style) Σ-language*, "(propositional/sentential/Hilbert-style) Σ-" standing for "L_Σ-". Likewise, to avoid appearance of redundant double subscripts, we normally use the subscript $_\Sigma$ alone for the double one $_{L_\Sigma}$, unless any confusion is possible. Any $\$^n(L_\Sigma)$-sequent, where $\omega \ni n = | \neq 0$, $\Phi = (\Gamma \vdash \Delta)$ is identified with the first-order equality-free clause|"quantifier-free formula" $(\bigwedge \Gamma) \to (\bigvee \Delta)$ of the signature $\Sigma \cup \{D\}$ under the identification of any Σ-term φ with the first-order atomic formula $D(\varphi)$ of the signature involved. In that case, sequent subsuming fits clause one adopted in [25], while sequent disjunction/implication is logically equivalent to formula disjunction/"implication under identification of any finite set of first-order formulas with its conjunction". Likewise, we get the notion of Φ's being *true*‖*satisfied in* any Σ-matrix $\mathcal{A}$ (*under* any $h \in \hom(\mathfrak{Tm}_\Sigma, \mathfrak{A})$) which fits that adopted in [17, 19, 30], and so the one of a *model of* any set $\mathcal{S}$ of $\$^n(L_\Sigma)$-sequents, the class of all them being denoted by $\mathrm{Mod}(\mathcal{S})$. And what is more, $\mathrm{Var} : \mathrm{Seq}_{\$^n(L_\Sigma)} \to \wp_\omega(\mathrm{Var}_\omega), (\Gamma \vdash \Delta) \mapsto (\bigcup \mathrm{Var}[\Gamma \cup \Delta])$ assigns finite sets of free variables of the first-order equality-free clauses|"quantifier-free formulas" identified with $\$^n(L_\Sigma)$-sequents.

Lemma 3.2. *Any multiplicative sequent Σ-calculus $\mathcal{G}$ is deductively multiplicative.*

Proof. Consider any $\mathcal{R} \in \mathcal{G}$, any $\sigma \in \mathrm{Sb}_\Sigma$ and any $\Phi = (\Gamma \vdash \Delta) \in \mathrm{Seq}_\Sigma$. Let $(m|n) \triangleq |\Gamma|\Delta|$. Take any enumeration $\mathbf{\Gamma}|\mathbf{\Delta}$ of $\Gamma|\Delta$. Then, $V \triangleq \mathrm{Var}(\mathcal{R}) \in \wp_\omega(\mathrm{Var}_\omega)$, in which case $|\,\mathrm{Var}_\omega \setminus V| = |\,\mathrm{Var}_\omega| = \omega \supseteq (m+n)$, for ω is infinite, while each element of it is finite, and so there is some injective $\bar{v} \in (\mathrm{Var}_\omega \setminus V)^{m+n}$. Let $\Psi \triangleq (\bar{v}[m] \vdash \bar{v}[(m+n) \setminus m])$ and $\sigma' \in \mathrm{Sb}_\Sigma$ extend $(\sigma \restriction \mathrm{Var}_{\omega \setminus (m+n)}) \cup (\mathbf{\Gamma} \circ (\bar{v} \restriction m)^{-1}) \cup (\mathbf{\Delta} \circ (\bar{v} \restriction ((m+n) \setminus m))^{-1})$. Then, $(\uplus\Psi)(\mathcal{R})$ is derivable in $\mathcal{G}$, for this is multiplicative, while $\sigma'(\mathcal{R}) = \sigma(\mathcal{R})$, whereas $\sigma'(\Psi) = \Phi$, in which case $(\uplus\Phi)(\sigma(\mathcal{R})) = \sigma'((\uplus\Psi)(\mathcal{R}))$ is derivable in $\mathcal{G}$, and so induction on the length of $\mathcal{G}$-derivations completes the argument. □

Clearly, every element of $\mathcal{S}_{\$^{[n+1]}(L_\Sigma)}$ [where $n \in (\omega \setminus 1)$] is true in any Σ-matrix $\mathcal{A}$, and so is that of $\mathcal{S}^{2\setminus 1}_{\$^{[n+1]}(L_\Sigma)}$, in which case, given a class of Σ-matrices M, the set $\mathcal{L}^{[n]}_{\mathsf{M}}$ of all $\$^{0[+n]}(L_\Sigma)$-rules true in M is a [deductively multiplicative] $\$^{0[+n]}(L_\Sigma)$-logic called the *[n-order sequent] logic of/"defined by"* M (cf. [8] for the non-optional case with one-element M), the reservation "n-order" being omitted, whenever $n = 1$, unless any confusion is possible. Then, the class of all "isomorphic copies"/"[consistent] submatrices" of members of M is denoted by $\mathbf{I}/\mathbf{S}_{[*]}(\mathsf{M})$, respectively, any class of Σ-matrices $\mathsf{K}[\subseteq \mathsf{M}]$ being said to be *[(*M*-)relatively] abstract/hereditary*, whenever $(\mathbf{I}/\mathbf{S}(\mathsf{K})[\cap \mathsf{M}]) \subseteq \mathsf{K}$, respectively. Likewise, M is said to be *[ultra-]multiplicative (up to isomorphisms)*, whenever every [ultra-]product of each tuple constituted by members of M is (isomorphic to) a member of M (i.e., $\mathbf{I}(\mathsf{M})$ is [ultra-]multiplicative). Clearly, any [abstract] class is (ultra-)multiplicative [if and] only if it is so up to isomorphisms. And what is more, the class of models of any $\$^{[0\cdot]n}(L_\Sigma)$-calculus, being a universal [strict Horn] first-order model class, is well-known to be both abstract, hereditary and ultra-multiplicative [as well as multiplicative] (cf., e.g., [10]). Likewise, any finite class of finite Σ-matrices is well-known to be ultra-multiplicative up to isomorphisms (cf., e.g., Corollary 2.3 of [5] for the purely-algebraic case immediately extended to the general one of algebraic systems).

Lemma 3.3. *Let* M *be a class of* Σ*-matrices, while [*$n \in (\omega \setminus 1)$*, whereas]* $\mathcal{A} \subseteq \mathrm{Fm}_{\$^{0[+n]}(L_\Sigma)}$*. Suppose* M *is ultra-multiplicative up to isomorphisms (in particular, both it and all members of it are finite). Then, the axiomatic extension* $\mathcal{L}'$ *of the [n-order sequent] logic* $\mathcal{L}$ *of* M *relatively axiomatized by* $\mathcal{A}$ *is defined by* $\mathsf{M}' \triangleq (\mathbf{S}(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A}))$*.*

Proof. Clearly, $\mathsf{M}' \subseteq \mathrm{Mod}(\mathcal{L} \cup \mathcal{A}) = \mathrm{Mod}(\mathcal{L}')$, for $\mathrm{Mod}(\mathcal{L}) \supseteq \mathsf{M}$ is hereditary. Conversely, consider any $\$^{0[+n]}(L_\Sigma)$-rule $\Phi \notin \mathcal{L}$, in which case, by (3.1), for each $X \in \wp_\omega(\mathrm{Sb}_\Sigma[\mathcal{A}])$, there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Tm}_\Sigma, \mathfrak{A})$ such that $\mathcal{A} \not\models \Phi[h]$, while, for all $\Psi \in X$, $\mathcal{A} \models \Psi[h]$, and so, by Mal'cev-Łoś Compactness Theorem factually for ultra-multiplicative up to isomorphisms classes of algebraic systems (cf., e.g., [10]), there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{Tm}_\Sigma, \mathfrak{A})$ such that $\mathcal{A} \not\models \Phi[h]$, while, for all $\Psi \in \mathrm{Sb}_\Sigma[\mathcal{A}]$, $\mathcal{A} \models \Psi[h]$. Then, $\mathcal{B} \triangleq (\mathcal{A} \restriction (\mathrm{img}\, h)) \in \mathbf{S}(\mathsf{M})$, while $h \in \hom(\mathfrak{Tm}_\Sigma, \mathfrak{B})$ is surjective, whereas $\mathcal{B} \not\models \Phi[h]$. Consider any $\Upsilon \in \mathcal{A}$ and any $g \in \hom(\mathfrak{Tm}_\Sigma, \mathfrak{B})$, in which case there is some $\sigma \in \mathrm{Sb}_\Sigma$ such that $g = (h \circ \sigma)$, and so $\mathcal{A} \models \sigma(\Upsilon)[h]$, that is, $\mathcal{B} \models \Upsilon[g]$ (in particular, $\mathcal{B} \in \mathsf{M}'$, as required). □

4. Preliminary issues

From now on, we fix any algebraic signature Σ as well as any $\varepsilon : \wp_\omega(\mathrm{Fm}_\Sigma) \to \mathrm{Fm}^*_\Sigma$ such that, for each $\Gamma \in \wp_\omega(\mathrm{Fm}_\Sigma)$, $\varepsilon(\Gamma)$ is an enumeration of Γ.

4.1. **Disjunctivity.** From now on, we fix a (possibly, secondary) binary connective $\veebar$ of Σ.

Let $\mathcal{G}^{\alpha}_{\veebar}$, where $1 \in \alpha \subseteq \omega$, be the α-conclusion sequent Σ-calculus constituted by the following α-conclusion sequent Σ-rules:

	Left	Right
Disjunctivity	$\dfrac{\{(\Gamma \cup \{x_0\}) \vdash \Delta; (\Gamma \cup \{x_1\}) \vdash \Delta\}}{(\Gamma \cup \{(x_0 \veebar x_1)\}) \vdash \Delta}$	$x_i \vdash (x_0 \veebar x_1)$

where $i \in 2$, while $\Gamma \in \wp_\omega(\mathrm{Var}_\omega)$, whereas $\Delta \in \wp_\alpha(\mathrm{Var}_\omega)$, in which case

$$(x_0 \veebar x_1) \quad \vdash \quad (x_1 \veebar x_0), \tag{4.1}$$

$$(x_0 \veebar x_1) \quad \vdash \quad x_0, \tag{4.2}$$

$$((x_0 \veebar x_1) \veebar x_2) \quad \dashv\vdash \quad (x_0 \veebar (x_1 \veebar x_2)). \tag{4.3}$$

are derivable in $\mathcal{G}^{\alpha}_{\veebar} \cup \mathcal{NS}^{\alpha}_{\varnothing}$, any $\veebar$-disjunctive Σ-matrix being a model of it. Then, a Σ-logic is said to be $\veebar$-*disjunctive*, whenever it contains Right Disjunctivity Σ-rules and is closed under all Σ-substitutional instances of Left Disjunctivity purely-single-conclusion sequent Σ-rules.

4.1.1. *Disjunctivity versus multiplicativity.* Likewise, a Σ-logic is said to be *[β-]$\veebar$-multiplicative* [where $\beta \subseteq \omega$], provided it is closed under

$$\frac{([\Gamma \cup]\Delta) \vdash \phi}{([\Gamma \cup](\veebar\psi)[\Delta] \vdash (\phi \veebar \psi)}, \tag{4.4}$$

where $\Delta \in \wp_{\omega[\cap\beta]}(\mathrm{Fm}_\Sigma)$ and $\phi, \psi \in \mathrm{Fm}_\Sigma$ [as well as $\Gamma \in \wp_\omega(\mathrm{Fm}_\Sigma)$]. Let $(\mathcal{A})\mathcal{M}^{[\gamma]}_{j,\veebar}$, where $j \in 2$ [and $\gamma \subseteq \omega$], be the purely-single-conclusion sequent Σ-calculus resulted from $\mathcal{NS}^{2\setminus 1}_{\varnothing}$ by adding Right Disjunctivity with $i = j$, (4.1), (4.2) and the [non-]non-optional version of (4.4) [with $\beta = \gamma$] (as well as (4.3)).

Lemma 4.1. *Let $j \in 2$ [and $\gamma \subseteq \omega$]. Then, any of rules of either of the calculi $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \mathcal{NS}^{2\setminus 1}_{\varnothing}$ or $(\mathcal{A})\mathcal{M}^{[\gamma]}_{j,\veebar}$ is derivable in another one. In particular, any Σ-logic is $\veebar$-disjunctive iff it both contains Right Disjunctivity with $i = j$,* (4.1) *and* (4.2) *(as well as* (4.3)*), and is [γ-]$\veebar$-multiplicative {that is ⟨in the "[]"-non-optional case⟩, for any Hilbert-style axiomatization $\mathcal{C}$ of it, each $\mathcal{R} \in \mathcal{C}$, every Σ-substitution σ and all $\varphi \in \mathrm{Tm}_\Sigma$, $(\veebar\varphi)(\sigma(\mathcal{R}))$ is derivable in $\mathcal{C}$}.*

Proof. First, we prove the derivability of the optional version of (4.4) with $\beta = \omega$ in $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \mathcal{NS}^{2\setminus 1}_{\varnothing}$ by induction on $n \triangleq |\Delta| \in \omega$. The case, when $\Delta = \varnothing$, is by Cut and Right Disjunctivity with $i = 0$. Otherwise, there is some $\varphi \in \Delta$, in which case $\Theta \triangleq (\Delta \setminus \{\varphi\}) \in \wp_n(\mathrm{Fm}_\Sigma)$, and so, by the induction hypothesis, the optional version of (4.4) with $\beta = \omega$ but with $(\Gamma \cup \{\varphi\})|\Theta$ instead of $\Gamma|\Delta$, respectively, is derivable in $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \mathcal{NS}^{2\setminus 1}_{\varnothing}$. And what is more, by Reflexivity, Right Disjunctivity with $i = 1$ and basic native structural rules, $(\Gamma \cup (\veebar\psi)[\Theta] \cup \{\psi\}) \vdash (\phi \veebar \psi)$ is derivable in $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \mathcal{NS}^{2\setminus 1}_{\varnothing}$. Hence, by Left Disjunctivity, the optional version of (4.4) with $\beta = \omega$ as such is derivable in $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \mathcal{NS}^{2\setminus 1}_{\varnothing}$, and so is [that with $\beta = \gamma$, for $\gamma \subseteq \omega$, as well as] the non-optional one, when taking $\Gamma = \varnothing$.

Conversely, by Right Disjunctivity with $i = j$, Cut and (4.1), Right Disjunctivity with $i = (1 - j)$ is derivable in $\mathcal{M}^{[\gamma]}_{j,\veebar}$. Then, by Right Disjunctivity with $i = 0 \in 2 = \{j, 1 - j\}$, applying Cut and basic native structural rules $|\Gamma|$ times, we see that $\dfrac{(\veebar\psi)[\Gamma] \cup (\veebar\psi)[\Delta]) \vdash (\phi \veebar \psi)}{(\Gamma \cup (\veebar\psi)[\Delta]) \vdash (\phi \veebar \psi)}$ is derivable in $\mathcal{M}_{j,\veebar}$ in which case, by the non-optional version of (4.4) but with $\Gamma \cup \Delta$ instead of Δ, the optional one with $\beta = (\omega[\cap\gamma])$ is derivable in $\mathcal{M}_{j,\veebar}$, and so it is derivable in $\mathcal{M}^{[\gamma]}_{j,\veebar}$. And what is more, by Cut, (4.1) and the optional version of (4.4) with $\beta = (\omega[\cap\gamma])$, $\Delta = x_0$, $\psi = x_1$, $\Gamma \in \wp_\omega(\mathrm{Var}_\omega)$ and $\phi \in \mathrm{Var}_\omega$, $\frac{(\Gamma \cup \{x_0\}) \vdash \phi}{\langle \Gamma, x_0 \veebar x_1 \rangle \vdash (x_1 \veebar \phi)}$ is derivable in $\mathcal{M}^{[\gamma]}_{j,\veebar}$. Likewise, by Cut, (4.2) and (4.4) with $\Delta = x_1$, $\psi = \phi$ and the same $\beta|\Gamma|\phi$, $\frac{(\Gamma \cup \{x_1\}) \vdash \phi}{(\Gamma \cup \{x_1 \veebar \phi\}) \vdash \phi}$ is derivable in $\mathcal{M}^{[\gamma]}_{j,\veebar}$. Thus, by basic native structural rules and Cut, Left Disjunctivity is derivable in $\mathcal{M}^{[\gamma]}_{j,\veebar}$, as required. □

Given any $\Phi = (\Gamma \vdash [\phi]) \in \mathrm{Seq}^{2[\backslash 1]}_\Sigma$, where $\Gamma \in \wp_\omega(\mathrm{Fm}_\Sigma)$ [and $\phi \in \mathrm{Fm}_\Sigma$], and any $\psi \in \mathrm{Fm}_\Sigma$, set $(\veebar\psi)^{\backslash 1}(\Phi) \triangleq ((\veebar\psi)[\Gamma] \vdash ([\phi\veebar]\psi))[= (\veebar\psi)(\Phi)] \in \mathrm{Seq}^{2\backslash 1}_\Sigma$.

Lemma 4.2. *Let $\mathcal{L}$ be a Σ-logic, $\Phi \in \mathrm{Seq}^2_\Sigma$, $\psi \in \mathrm{Fm}_\Sigma$, $\sigma \in \mathrm{Sb}_\Sigma$ and $v \in (\mathrm{Var}_\omega \setminus \mathrm{Var}(\Phi))$. Suppose $\mathcal{L}$ contains both* (4.3) *and $\mathcal{R} \triangleq (\veebar v)^{\backslash 1}(\Phi)$. Then, it contains $(\veebar\psi)(\sigma(\mathcal{R}))$.*

Proof. Let $\sigma' \in \mathrm{Sb}_\Sigma$ extend $(\sigma\restriction(\mathrm{Var}_\omega \setminus \{v\})) \cup [v/(\sigma(v) \veebar \psi)]$, in which case $\sigma(\Phi) = \sigma'(\Phi)$, for $v \notin \mathrm{Var}(\Phi)$, and so $\mathcal{L}$ contains $\sigma'(\mathcal{R}) = (\veebar(\sigma(v) \veebar \psi))^{\backslash 1}(\sigma(\Phi))$, (in particular, by (4.3), it contains $(\veebar\psi)((\veebar\sigma(v))^{\backslash 1}(\sigma(\Phi))) = (\veebar\psi)(\sigma(R))$, as required). □

Let $\sigma_{+m} \in \hom(\mathfrak{Tm}_\Sigma, \mathfrak{Tm}_\Sigma)$, where $m \in \omega$, extend $[x_i/x_{i+m}]_{i\in\omega}$. Given any $\mathcal{S} \subseteq \mathrm{Seq}^{[2]}_\Sigma$, put $\mathcal{S}^{\backslash 1} \triangleq ((\mathcal{S} \cap \mathrm{Seq}^{\omega\backslash 1}_\Sigma) \cup \{(\sigma_{+1}[\Gamma] \vdash x_0) \mid \Gamma \in \mathrm{Fm}^*_\Sigma, (\Gamma\vdash) \in \mathcal{S}\}) \subseteq \mathrm{Seq}^{\omega\backslash 1}_\Sigma$ [and $\mathfrak{R}_\veebar(\mathcal{S}) \triangleq ((\mathcal{S} \cap ((\mathrm{Fm}^0_\Sigma \times \mathrm{Fm}^1_\Sigma) \cup \bigcup_{j\in\omega}(\mathrm{Var}^1_{\{j\}} \times (\mathrm{Tm}^{\omega\backslash\{j\}}_\Sigma)^1))) \cup (\veebar x_0)^{\backslash 1}[\sigma_{+1}[\mathcal{S} \setminus ((\mathrm{Fm}^0_\Sigma \times \mathrm{Fm}^1_\Sigma) \cup \bigcup_{j\in\omega,k\in 2}(\mathrm{Var}^1_{\{j\}} \times (\mathrm{Tm}^{\omega\backslash\{j\}}_\Sigma)^k))]] \cup \{x_{j+1} \vdash x_0 \mid j \in \omega, (x_j \vdash) \in \mathcal{S}\}) \subseteq \mathrm{Seq}^{2\backslash 1}_\Sigma$].

Corollary 4.3. *Let $\mathcal{L}$ be a $\veebar$-disjunctive Σ-logic, $\mathcal{S} \subseteq \mathrm{Seq}^2_\Sigma$ and $\mathcal{L}'$ the extension of $\mathcal{L}$ relatively axiomatized by $\mathfrak{R}_\veebar(\mathcal{S})$. Then, the following hold:*

(i) *$\mathcal{L}'$ is $\veebar$-disjunctive;*
(ii) *$\mathcal{S}^{\backslash 1} \subseteq \mathcal{L}'$.*

In particular, any axiomatic extension of any $\veebar$-disjunctive Σ-logic is $\veebar$-disjunctive.

Proof. (i) is proved with applying the "()"-optional "[]"-non-optional version of the "only if" {resp., "if"} part of the second assertion of Lemma 4.1 with $j = 0$ and $\mathcal{C} = (\mathcal{L}\{\cup\mathfrak{R}_\veebar(\mathcal{S})\})$ to $\mathcal{L}\{'\}$, respectively. For consider any Σ-substitution σ and any $\varphi \in \mathrm{Fm}_\Sigma$. Then, for any $\phi \in \mathrm{Fm}_\Sigma$ such that $(\varnothing \vdash \phi) \in \mathcal{S}$, $\phi \in \mathfrak{R}_\veebar(\mathcal{S}) \subseteq (\mathcal{L} \cup \mathfrak{R}_\veebar(\mathcal{S}))$, in which case $\sigma(\phi)$ is derivable in $\mathcal{L} \cup \mathfrak{R}_\veebar(\mathcal{S})$, and so is $\sigma(\phi) \veebar \varphi$, in view of Right Disjunctivity with $i = 0$. And what is more, for any $j \in \omega$ and any $\phi \in \mathrm{Tm}^{\omega\backslash\{j\}}_\Sigma$ such that $\mathcal{R} = (x_j \vdash \phi) \in \mathcal{S}$, $\mathcal{R} \in \mathfrak{R}_\veebar(\mathcal{S}) \subseteq (\mathcal{L} \cup \mathfrak{R}_\veebar(\mathcal{S}))$, in which case $\sigma'(\mathcal{R}) = ((\sigma(x_j) \veebar \varphi) \vdash \sigma(\phi))$, where $\sigma' \in \mathrm{Sb}_\Sigma$ extends $(\sigma\restriction \mathrm{Var}_{\omega\backslash\{j\}}) \cup [x_j/(\sigma(x_j) \veebar \varphi)]$, is derivable in $\mathcal{L} \cup \mathfrak{R}_\veebar(\mathcal{S})$, and so is $(\veebar\varphi)(\sigma(\mathcal{R}))$, in view of Right Disjunctivity with $i = 0$. Likewise, for any $j \in \omega$ such that $(x_j \vdash) \in \mathcal{S}$, $\mathcal{R} \triangleq (x_{j+1} \vdash x_0) \in \mathfrak{R}_\veebar(\mathcal{S}) \subseteq (\mathcal{L} \cup \mathfrak{R}_\veebar(\mathcal{S}))$, in which case $\sigma''(\mathcal{R}) = ((\sigma(x_{j+1}) \veebar \varphi) \vdash (\sigma(x_0) \veebar \varphi)) = (\veebar\varphi)(\sigma(\mathcal{R}))$, where $\sigma'' \in \mathrm{Sb}_\Sigma$ extends $[x_l/(\sigma(x_l) \veebar \varphi)]_{l\in\omega}$, is derivable in $\mathcal{L} \cup \mathfrak{R}_\veebar(\mathcal{S})$. In this way, Lemma 4.2 with $v = x_0$ and $\psi = \varphi$ completes the argument.

(ii) Consider any $\Phi \in \mathcal{S}$. Then, in case $\Phi \in ((\mathrm{Fm}^0_\Sigma \times \mathrm{Fm}^1_\Sigma) \cup \bigcup_{j\in\omega}(\mathrm{Var}^1_{\{j\}} \times (\mathrm{Tm}^{\omega\backslash\{j\}}_\Sigma)^1)) \subseteq \mathrm{Seq}^{2\backslash 1}_\Sigma$, $\Phi^{\backslash 1} = \Phi \in \mathfrak{R}_\veebar(\mathcal{S}) \subseteq \mathcal{L}'$. Likewise, in case $\Phi = (x_j \vdash)$, for some $j \in \omega$, $\Phi^{\backslash 1} \in \mathfrak{R}_\veebar(\mathcal{S}) \subseteq \mathcal{L}'$. Otherwise, $\Phi = (\Gamma \vdash [\varphi])$, for some $\Gamma \in \wp_{\omega\backslash 1}(\mathrm{Fm}_\Sigma)$ [and some $\varphi \in \mathrm{Fm}_\Sigma$], in which case $((\veebar x_0)[\sigma_{+1}[\Gamma]] \vdash ([\sigma_{+1}(\varphi)\veebar]x_0)) \in \mathfrak{R}_\veebar(\mathcal{S}) \subseteq \mathcal{L}'$, and so, by (i) and Right Disjunctivity with $i = 0$, $(\sigma_{+1}[\Gamma] \vdash ([\sigma_{+1}(\varphi)\veebar]x_0)) \in \mathcal{L}'$. [Let $\sigma' \in \mathrm{Sb}_\Sigma$ extend $[x_0/\varphi; x_{i+1}/x_i]_{i\in\omega}$, in which case $(\Gamma \vdash (\varphi \veebar \varphi)) = \sigma'(\sigma_{+1}[\Gamma] \vdash (\sigma_{+1}(\varphi) \veebar x_0)) \in \mathcal{L}'$, and so, by (i) and Lemma 4.1(4.2), $(\Gamma \vdash \varphi) \in \mathcal{L}'$.] Thus, in any case, $\Phi^{\backslash 1} \in \mathcal{L}'$, as required. □

4.1.2. *Single- and purely- versus multi-conclusion sequent calculi.* Let

$$\tau_{\veebar} : \mathrm{Seq}_{\Sigma} \to \mathrm{Seq}^{2}_{\Sigma}, (\Gamma \vdash \Delta) \mapsto \begin{cases} \Gamma \vdash \Delta & \text{if } \Delta = \varnothing, \\ \Gamma \vdash (\veebar\varepsilon(\Delta)) & \text{otherwise.} \end{cases}$$

Then,

$$\sigma(\tau_{\veebar}(\Phi)) = \tau_{\veebar}(\sigma(\Phi)), \tag{4.5}$$

for all $\Phi \in \mathrm{Seq}_{\Sigma}$ and all $\sigma \in \mathrm{Sb}_{\Sigma}$.

Lemma 4.4. *Let $\psi \in \mathrm{Tm}_{\veebar}$, $v \in \mathrm{Var}(\psi)$ and $\alpha \in \wp(2 \setminus 1, \omega)$. Then, $v \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar} \cup \mathcal{NS}^{\alpha}_{\varnothing}$.*

Proof. By induction on construction of ψ. For consider the following complementary cases:

(1) $\psi \in \mathrm{Var}_{\omega}$.
Then, $\mathrm{Var}(\psi) = \{\psi\} \ni v$, in which case $\psi = v$, and so Reflexivity completes the argument.

(2) $\psi \notin \mathrm{Var}_{\omega}$.
Then, $\psi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Tm}_{\veebar}$, in which case $v \in \mathrm{Var}(\psi) = (\bigcup_{j \in 2} \mathrm{Var}(\varphi_j))$, and so $v \in \mathrm{Var}(\varphi_j)$, for some $j \in 2$. Hence, by induction hypothesis, $v \vdash \varphi_j$ is derivable in $\mathcal{G}^{\alpha}_{\veebar} \cup \mathcal{NS}^{\alpha}_{\varnothing}$. In this way, Cut and Right Disjunctivity with $i = j$ complete the argument. □

Corollary 4.5. *Let $\phi, \psi \in \mathrm{Tm}_{\veebar}$ and $\alpha \in \wp(2 \setminus 1, \omega)$. Suppose $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$. Then, $\phi \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar} \cup \mathcal{NS}^{\alpha}_{\varnothing}$.*

Proof. By induction on construction of ϕ. For consider the following complementary cases:

(1) $\phi \in \mathrm{Var}_{\omega}$.
Then, $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = \{\phi\}$, in which case $\phi \in \mathrm{Var}(\psi)$, and so Lemma 4.4 completes the argument.

(2) $\phi \notin \mathrm{Var}_{\omega}$.
Then, $\phi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Tm}_{\veebar}$, in which case $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = (\bigcup_{j \in 2} \mathrm{Var}(\varphi_j))$, and so $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\varphi_j)$, for each $j \in 2$. Hence, by induction hypothesis, $\varphi_j \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar} \cup \mathcal{NS}^{\alpha}_{\varnothing}$, for every $j \in 2$. In this way, Left Disjunctivity completes the argument. □

Theorem 4.6. *For every $\mathcal{R} \in \mathcal{G}^{\omega[\setminus 1]}_{\veebar} \cup \mathcal{NS}^{\omega[\setminus 1]}_{\varnothing}$, $\tau_{\veebar}(\mathcal{R})$ is derivable in $\mathcal{G}^{2[\setminus 1]}_{\veebar} \cup \mathcal{NS}^{2[\setminus 1]}_{\varnothing}$. In particular, for all $(\mathcal{S} \cup \{\Phi\}) \subseteq \mathrm{Seq}^{\omega[\setminus 1]}_{\Sigma}$ such that Φ is derivable in $\mathcal{G}^{\omega[\setminus 1]}_{\veebar} \cup \mathcal{NS}^{\omega[\setminus 1]}_{\varnothing} \cup \mathcal{S}$, $\tau_{\veebar}(\Phi)$ is derivable in $\mathcal{G}^{2[\setminus 1]}_{\veebar} \cup \mathcal{NS}^{2[\setminus 1]}_{\varnothing} \cup \tau_{\veebar}[\mathcal{S}]$.*

Proof. Consider the following exhaustive cases:

(1) $\mathcal{R}$ is either in $\mathcal{G}^{\omega[\setminus 1]}_{\veebar}$ or Reflexivity or an instance of Cut with $\Delta = \varnothing$,
in which case $\tau_{\veebar}(\mathcal{R})$ is a Σ-substitutional instance of a rule in $\mathcal{G}^{2[\setminus 1]}_{\veebar} \cup \mathcal{NS}^{2[\setminus 1]}_{\varnothing}$, and so is derivable in it.

(2) $\mathcal{R}$ is an instance of Diagonal Subsuming,
in which case $\tau_{\veebar}(\mathcal{R})$ is of the form $\dfrac{\Lambda \vdash \phi}{\Theta \vdash \psi}$, where $\Lambda, \Theta \in \wp_{\omega}(V_{\omega})$ and $\phi, \psi \in \mathrm{Tm}_{\veebar}$ such that $(\Lambda \subseteq \Theta$ and $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, and so Corollary 4.5 as well as both Diagonal Subsuming and Cut complete the argument of the first assertion.

(3) $\mathcal{R}$ is an instance of Cut with $\Delta \neq \varnothing$.
Then, $\tau_{\veebar}(\mathcal{R})$ is of the form $\dfrac{\{(\Lambda \cup \Gamma) \vdash \varphi, (\Gamma \cup \{v\}) \vdash \psi\}}{(\Lambda \cup \Gamma) \vdash \psi}$, where $v \in \mathrm{Var}_{\omega}$, $\varphi \triangleq (\veebar\varepsilon(\Delta \cup \{v\})) \in \mathrm{Tm}_{\veebar}$, $\phi \triangleq (\veebar\varepsilon(\Delta)) \in \mathrm{Tm}_{\veebar}$ and $\psi \triangleq (\veebar\varepsilon(\Delta \cup \Theta)) \in$

$\mathrm{Tm}_{\veebar}$, in which case $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, while $(\phi \veebar v) \in \mathrm{Tm}_\Sigma$, whereas $\mathrm{Var}(\phi \veebar v) = \mathrm{Var}(\varphi)$, and so, by Corollary 4.5, both $\phi \vdash \psi$ and $\varphi \vdash (\phi \veebar v)$ are derivable in $\mathcal{G}_{\veebar}^{2[\backslash 1]} \cup \mathcal{NS}_{\varnothing}^{2[\backslash 1]}$, and so are both $(\Gamma \cup \{\phi\}) \vdash \psi$ and $\frac{(\Lambda \cup \Gamma) \vdash \varphi}{(\Lambda \cup \Gamma) \vdash (\phi \veebar v)}$, by Diagonal Subsuming and Cut, respectively. In particular, by Left Disjunctivity, the rule $\frac{(\Gamma \cup \{v\}) \vdash \psi}{(\Gamma \cup \{\phi \veebar v\}) \vdash \psi}$ is derivable in $\mathcal{G}_{\veebar}^{2[\backslash 1]} \cup \mathcal{NS}_{\varnothing}^{2[\backslash 1]}$. In this way, Cut completes the argument of the first assertion.

Finally, the second assertion is by the first one, the induction on the length of $(\mathcal{G}_{\veebar}^{\omega[\backslash 1]} \cup \mathcal{NS}_{\varnothing}^{\omega[\backslash 1]} \cup \mathcal{S})$-derivations and (4.5). $\square$

Lemma 4.7. *Let $\mathcal{S} \subseteq \mathrm{Seq}_\Sigma$. Then, any purely-multi-conclusion Σ-sequent is derivable in $\mathcal{G}_{\veebar}^{\omega\backslash 1} \cup \mathcal{NS}_{\varnothing}^{\omega\backslash 1} \cup \mathcal{S}^{\backslash 1}$, whenever it is derivable in $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{S}$.*

Proof. Consider any $\Phi = (\Gamma \vdash \Delta) \in \mathrm{Seq}_\Sigma^{\omega\backslash 1}$ derivable in $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{S}$. Take any $\varphi \in \Delta \neq \varnothing$. Clearly, $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing}$ is multiplicative, and so deductively so, in view of Lemma 3.2. In particular, for any Σ-substitutional instance $\mathcal{R}$ of any rule in it, $(\uplus(\vdash \varphi))(\mathcal{R})$ is derivable in it, and so, being purely-multi-conclusion, in $\mathcal{G}_{\veebar}^{\omega\backslash 1} \cup \mathcal{NS}_{\varnothing}^{\omega\backslash 1} \cup \mathcal{S}^{\backslash 1}$. Now, consider any $\Psi = (\Lambda \vdash \Theta) \in \mathcal{S}$ and any $\sigma \in \mathrm{Sb}_\Sigma$. If $\Theta \neq \varnothing$, then $\Psi \in \mathcal{S}^{\backslash 1}$, in which case $\sigma(\Psi) \preceq_{\eth} (\sigma(\Psi) \uplus (\vdash \varphi))$ is derivable in $\mathcal{G}_{\veebar}^{\omega\backslash 1} \cup \mathcal{NS}_{\varnothing}^{\omega\backslash 1} \cup \mathcal{S}^{\backslash 1}$, and so is $\sigma(\Psi) \uplus (\vdash \varphi)$, by Diagonal Subsuming. Otherwise, $\Upsilon \triangleq (\sigma_{+1}(\Lambda) \vdash x_0) \in \mathcal{S}^{\backslash 1}$, in which case $(\sigma(\Psi) \uplus (\vdash \varphi)) = \sigma'(\Upsilon)$, where $\sigma' \in \mathrm{Sb}_\Sigma$ extends $[x_0/\varphi; x_{i+1}/\sigma(x_i)]_{i \in \omega}$, is derivable in $\mathcal{G}_{\veebar}^{\omega\backslash 1} \cup \mathcal{NS}_{\varnothing}^{\omega\backslash 1} \cup \mathcal{S}^{\backslash 1}$, and so, by induction on the length of $(\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{S})$-derivations, we conclude that $(\Phi \uplus (\vdash \varphi) \preceq_{\eth} \Phi$ is derivable in $\mathcal{G}_{\veebar}^{\omega\backslash 1} \cup \mathcal{NS}_{\varnothing}^{\omega\backslash 1} \cup \mathcal{S}^{\backslash 1}$ (in particular, Φ is so, by Diagonal Subsuming). $\square$

Corollary 4.8. *Let $\mathcal{S} \subseteq \mathrm{Seq}_\Sigma$. Then, any [purely-]single-conclusion Σ-sequent is derivable in $\mathcal{G}_{\veebar}^{2[\backslash 1]} \cup \mathcal{NS}_{\varnothing}^{2[\backslash 1]} \cup \tau_{\veebar}[\mathcal{S}^{[\backslash 1]}]$, whenever it is derivable in $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{S}$.*

Proof. By Theorem 4.6 [and Lemma 4.7], for $\tau_{\veebar} \restriction \mathrm{Seq}_\Sigma^{2[\backslash 1]}$ is diagonal. $\square$

4.1.3. *The basic disjunctive Hilbert-style calculus.* By $\mathcal{B}_{\veebar}$ we denote the Σ-calculus constituted by the following Σ-rules:

$$\underset{B_1}{\frac{x_0 \veebar x_0}{x_0}} \qquad \underset{B_2}{\frac{x_0}{x_0 \veebar x_1}} \qquad \underset{B_3}{\frac{(x_0 \veebar x_1) \veebar x_2}{(x_1 \veebar x_0) \veebar x_2}} \qquad \underset{B_4}{\frac{(x_0 \veebar (x_1 \veebar x_2)) \veebar x_3}{((x_0 \veebar x_1) \veebar x_2) \veebar x_3}}$$

Lemma 4.9. *Let $\mathcal{L}$ be a Σ-logic, $\mathcal{R} = (\Gamma \vdash \phi)$ a Σ-rule and $v \in (\mathrm{Var} \setminus \mathrm{Var}(\mathcal{R}))$. Suppose $\mathcal{L}$ contains both Right Disjunctivity with $i = 0$ and (4.2) as well as $(\veebar v)(\mathcal{R})$. Then, $\mathcal{R} \in \mathcal{L}$.*

Proof. In that case, $\mathcal{L}$ contains $((\veebar v)(\mathcal{R})[v/\phi]) = (\veebar\phi)(\mathcal{R})$, and so $\Gamma \vdash (\phi \veebar \phi)$, in view of Right Disjunctivity with $i = 0$. In this way, (4.2) completes the argument. $\square$

Taking B_1 and B_2 into account and applying Lemma 4.9 with $\mathcal{L} = \mathcal{L}_{\mathcal{B}_{\veebar}}$ to both B_3 and B_4, we immediately get:

Corollary 4.10. *The following rules are derivable in $\mathcal{B}_{\veebar}$:*

$$\frac{x_0 \veebar x_1}{x_1 \veebar x_0}, \tag{4.6}$$

$$\frac{x_0 \veebar (x_1 \veebar x_2)}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.7}$$

Lemma 4.11. *The following rules are derivable in* $\mathcal{D}_{\veebar}$*:*

$$\frac{(x_0 \veebar x_1) \veebar x_2}{x_0 \veebar (x_1 \veebar x_2)}, \tag{4.8}$$

$$\frac{(x_0 \veebar x_0) \veebar x_1}{x_0 \veebar x_1}, \tag{4.9}$$

$$\frac{x_0 \veebar x_2}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.10}$$

Proof. First, in view of Corollary 4.10, (4.8) is by the following $\mathcal{L}_{\mathcal{B}_{\veebar}}$-derivation:

(1) $(x_0 \veebar x_1) \veebar x_2$ — hypothesis;
(2) $(x_1 \veebar x_0) \veebar x_2$ — B_3: 1;
(3) $x_2 \veebar (x_1 \veebar x_0)$ — (4.6)$[x_0/(x_1 \veebar x_0), x_1/x_2]$: 2;
(4) $(x_2 \veebar x_1) \veebar x_0$ — (4.7)$[x_0/x_2, x_2/x_0]$: 3;
(5) $(x_1 \veebar x_2) \veebar x_0$ — $B_3[x_0/x_2, x_2/x_0]$: 4;
(6) $x_0 \veebar (x_1 \veebar x_2)$ — (4.6)$[x_0/(x_1 \veebar x_0), x_1/x_0]$: 5.

Then, in view of Corollary 4.10, (4.9) is by the following $\mathcal{L}_{\mathcal{B}_{\veebar}}$-derivation:

(1) $(x_0 \veebar x_0) \veebar x_1$ — hypothesis;
(2) $x_0 \veebar (x_0 \veebar x_1)$ — (4.8)$[x_1/x_0, x_2/x_1]$: 1;
(3) $(x_0 \veebar x_1) \veebar x_0$ — (4.6)$[x_1/(x_0 \veebar x_1)]$: 2;
(4) $((x_0 \veebar x_1) \veebar x_0) \veebar x_1$ — $B_2[x_0/((x_0 \veebar x_1) \veebar x_0)]$: 3;
(5) $(x_0 \veebar x_1) \veebar (x_0 \veebar x_1)$ — (4.8)$[x_0/(x_0 \veebar x_1), x_1/x_0, x_1/x_2]$: 4;
(6) $(x_0 \veebar x_1)$ — $B_1[x_0/(x_0 \veebar x_1)]$: 5.

Finally, in view of Corollary 4.10, (4.10) is by the following $\mathcal{L}_{\mathcal{B}_{\veebar}}$-derivation:

(1) $x_0 \veebar x_2$ — hypothesis;
(2) $(x_0 \veebar x_2) \veebar x_1$ — $B_2[x_0/(x_0 \veebar x_2)]$: 1;
(3) $x_0 \veebar (x_2 \veebar x_1)$ — (4.8)$[x_1/x_2, x_2/x_1]$: 2;
(4) $(x_2 \veebar x_1) \veebar x_0$ — (4.6)$[x_1/(x_2 \veebar x_1)]$: 3;
(5) $x_2 \veebar (x_1 \veebar x_0)$ — (4.8)$[x_0/x_2, x_2/x_0]$: 4;
(6) $(x_1 \veebar x_0) \veebar x_2$ — (4.6)$[x_0/x_2, x_1/(x_1 \veebar x_0)]$: 5;
(7) $(x_0 \veebar x_1) \veebar x_2$ — $B_3[x_0/x_1, x_1/x_0]$: 6. □

Theorem 4.12. $\mathcal{L} \triangleq \mathcal{L}_{\mathcal{B}_{\veebar}}$ *is the least* $\veebar$*-disjunctive* Σ*-logic. In particular, each rule of* $\mathcal{B}_{\veebar}$ *is true in every* $\veebar$*-disjunctive* Σ*-matrix.*

Proof. Let $\mathcal{L}'$ be a $\veebar$-disjunctive Σ-logic, in which case, by the "()"-optional "[]"-non-optional version of Lemma 4.1, it is $\veebar$-multiplicative as well as contains Right Disjunctivity with $i = 0$ (viz., B_2), (4.1), (4.2) $= B_1$ and includes (4.3), in which case it contains|includes $(((\veebar x_2)(4.1))|(\veebar x_3)[4.3])(= | \ni)B_{3|4}$, and so is an extension of $\mathcal{L}$.

Finally, we prove the $\veebar$-disjunctivity of $\mathcal{L}$ with using the "()"-non-optional "[]"-non-optional version of Lemma 4.1 with $\mathcal{C} = \mathcal{B}_{\veebar}$. First, by B_1, B_2, Corollary 4.10 and Lemma 4.11(4.8), Right Disjunctivity with $i = 0$, (4.1), (4.2) and (4.3) are in $\mathcal{L}$.

Next, consider any $\sigma \in \mathrm{Sb}_\Sigma$, any $\psi \in \mathrm{Fm}_\Sigma$ and any $j \in (5 \setminus 1)$. The case, when $j \notin 3$, is due to Lemma 4.2 with $v = x_{j-1}$ and such $\mathcal{R}$ that $B_j = (\veebar v)(\mathcal{R})$. Otherwise, we have $\mathrm{Var}(B_j) = V_i \not\ni x_j$. Then, by Lemma 4.11(4.9)/(4.10), $(\veebar x_j)(B_j)$, where $j = (1/2)$, is derivable in $\mathcal{B}_{\veebar}$. Let $\sigma' \in \mathrm{Sb}_\Sigma$ extend $(\sigma \restriction V_{\omega \setminus \{j\}}) \cup [x_j/\psi]$, in which case $\sigma'(B_j) = \sigma(B_j)$, and so we eventually conclude that $(\veebar\psi)(\sigma(B_j)) = (\veebar\sigma'(x_j))(\sigma'(B_j)) = \sigma'((\veebar x_j)(B_j))$ is derivable in $\mathcal{B}_{\veebar}$, as required. □

The following auxiliary observation has proved quite useful for reducing the number of rules of calculi to be constructed in Section 7 according to the universal method to be elaborated in Section 6:

Corollary 4.13. *Let $\mathcal{C}[']$ be Σ-calculi, $\phi,\psi,\varphi \in \mathrm{Fm}_\Sigma$ and $v \in (\mathrm{Var}_\omega \setminus (\bigcup \mathrm{Var}[\{\phi,\psi,\varphi\}]))$. Suppose $\mathcal{L} \triangleq \mathcal{L}_{\mathcal{C}} \subseteq \mathcal{L}' \triangleq \mathcal{L}_{\mathcal{C}'}$ is $\veebar$-disjunctive (in particular, $\mathcal{C} = \mathcal{B}_{\veebar}$; cf. Theorem 4.12). Then, the rules $(\phi \veebar v) \vdash (\varphi \veebar v)$ and $(\psi \veebar v) \vdash (\varphi \veebar v)$ are both derivable in $\mathcal{C}'$ iff the rule $((\phi \veebar \psi) \vee v) \vdash (\varphi \vee v)$ is so.*

Proof. First of all, by the "()"-optional "[]"-non-optional version of Lemma 4.1, $\mathcal{L}$ is $\veebar$-multiplicative as well as contains both (4.1), (4.2) and (4.3). Then, the "if" part is by Right Disjunctivity with $i = (0/1)$ and the non-optional version of (4.4) with $\psi = v$ and $\Delta = (\phi/\psi)$, for $\mathcal{L} \subseteq \mathcal{L}'$. Conversely, assume both $(\phi \veebar v) \vdash (\varphi \vee v)$ and $(\psi \veebar v) \vdash (\varphi \vee v)$ are derivable in $\mathcal{C}'$, applying $[v/(\psi \veebar v)]$ and $[v/(v \veebar \varphi)]$, respectively, to which, we see that both $(\phi \veebar (\psi \veebar v)) \vdash (\varphi \veebar (\psi \veebar v))$ and $(\psi \veebar (v \veebar \varphi)) \vdash (\varphi \veebar (v \veebar \varphi))$ are derivable in $\mathcal{C}'$. In this way, as $\mathcal{L} \subseteq \mathcal{L}'$, by the non-optional version of (4.4) with $\psi = v$ and $\Delta = (\varphi \veebar \varphi)$ as well as (4.2), we have $(((\varphi \veebar \varphi) \veebar v) \vdash (\varphi \veebar v)) \in \mathcal{L}'$, in which case, by (4.1) and (4.3), we get the following $\mathcal{L}'$-derivation $\langle ((\phi \veebar \psi) \veebar v), (\phi \veebar (\psi \veebar v)), (\varphi \veebar (\psi \veebar v)), ((\psi \veebar v) \veebar \varphi), (\psi \veebar (v \veebar \varphi)), (\varphi \veebar (v \veebar \varphi)), ((v \veebar \varphi) \veebar \varphi), (v \veebar (\varphi \veebar \varphi)), ((\varphi \veebar \varphi) \veebar v), (\varphi \veebar v) \rangle$ of $(((\phi \veebar \psi) \vee v) \vdash (\varphi \vee v))$, and so this is derivable in $\mathcal{C}'$, as required. □

4.2. **Implicativity.** From now on, we fix any (possibly, secondary) binary connective $\sqsupset$ of Σ.

A Σ-logic is said to *have Deduction Theorem (DT) with respect to* $\sqsupset$, provided it is closed under all Σ-substitutional instances of the pure-single-conclusion sequent Σ-rule:

$$\frac{(\Gamma \cup \{x_0\}) \vdash x_1}{\Gamma \vdash (x_0 \sqsupset x_1)}, \tag{4.11}$$

where $\Gamma \in \wp_\omega(\mathrm{Var}_\omega)$. Then, a Σ-logic is said to be *[strongly] $\sqsupset$-implicative*, whenever it has DT with respect to $\sqsupset$ and contains [both] the *Modus Ponens* rule:

$$\{x_0, x_0 \sqsupset x_1\} \vdash x_1 \tag{4.12}$$

[and the *Peirce Law* axiom (cf. [13]):

$$(x_0 \sqsupset x_1) \veebar_{\sqsupset} x_0], \tag{4.13}$$

in which case it also contains:

$$x_0 \sqsupset (x_1 \sqsupset x_0) \tag{4.14}$$

$$(x_0 \sqsupset (x_1 \sqsupset x_2)) \sqsupset ((x_0 \sqsupset x_1) \sqsupset (x_0 \sqsupset x_2)), \tag{4.15}$$

$$x_0 \veebar_{\sqsupset} (x_0 \sqsupset x_1). \tag{4.16}$$

Let $\mathcal{I}[\mathcal{S}]^{(\mathrm{PL})}_{\sqsupset}$ be the [purely-single-conclusion sequent] Σ-calculus constituted by (4.12) and both (4.14) and (4.15) [resp., (4.11) and all native structural purely-single-conclusion sequent $\varnothing$-rules] (as well as (4.13)), each element of it being true in every $\sqsupset$-implicative {in particular, $\veebar_{\sqsupset}$-disjunctive} Σ-matrix. Then, using the well-known derivability of $x_0 \sqsupset x_0$ in $\mathcal{I}_{\sqsupset}$ as well as Herbrand's method (cf., e.g., the proof of Proposition 1.8 of [12]), we have:

Lemma 4.14. *Any axiomatic extension of $\mathcal{I}_{\sqsupset}$ has DT with respect to $\sqsupset$. In particular, [strongly] $\sqsupset$-implicative Σ-logics are exactly axiomatic extensions of $\mathcal{I}^{[\mathrm{PL}]}_{\sqsupset}$, in which case this is the least one, and so its rules are true in any $\sqsupset$-implicative Σ-matrix.*

4.2.1. *Implicativity versus disjunctivity.*

Lemma 4.15. *Let $\veebar \triangleq \veebar_{\sqsupset}$. Then, both the optional version of (4.4) with $\beta = (2\backslash 1)$ and Right Disjunctivity with $i = 1$ [as well as both (4.1) and (4.2)] are derivable in $\mathcal{IS}^{[\mathrm{PL}]}_{\sqsupset}$. In particular, any $\sqsupset$-implicative Σ-logic (i.e., an axiomatic extension of $\mathcal{I}_{\sqsupset}$; cf. Lemma 4.14)*

(i) *is* $(2 \setminus 1)$-$\veebar$-*multiplicative;*
(ii) *is* $\veebar$-*disjunctive iff it contains* (4.1)/(4.13) *iff it is strongly* $\sqsupset$-*implicative (i.e., an axiomatic extension of* $\mathcal{I}^{\mathrm{PL}}_{\sqsupset}$*; cf. Lemma 4.14).*

Proof. First, consider any $\Gamma \in \wp_\omega(\mathrm{Fm}_\Sigma)$ and any $\phi, \psi, \varphi \in \mathrm{Fm}_\Sigma$. Clearly,

$$\frac{(\Gamma \cup \{\phi\}) \vdash \psi}{(\Gamma \cup \{\psi \sqsupset \varphi\}) \vdash (\phi \sqsupset \varphi)} \tag{4.17}$$

is derivable in $\mathcal{IS}^{[\mathrm{PL}]}_{\sqsupset}$. Then, applying (4.17) once more but with $(\phi|\psi) \sqsupset \varphi$ instead of $\psi|\phi$, respectively, we see that the optional version of (4.4) with $\beta = (2 \setminus 1)$ is derivable in $\mathcal{IS}_{\sqsupset}$. Next, the derivability of Right Disjunctivity with $i = 1$ in $\mathcal{IS}_{\sqsupset}$ is by Reflexivity, Diagonal Subsuming and (4.11)$[x_0/(x_0 \sqsupset x_1)]$ with $\Gamma = x_1$. [Further, the derivability of (4.2) in $\mathcal{IS}^{\mathrm{PL}}_{\sqsupset}$ is by (4.13)$[x_1/x_0]$, (4.12)$[x_0/(x_0 \veebar x_0), x_1/x_0]$ and Cut. Finally, by (4.12)$[x_0/(x_1|(x_0 \sqsupset x_1)), x_1/x_{0|1}]$, both of $\{x_1|(x_0 \sqsupset x_1), (x_1 \sqsupset x_0)|(x_0 \veebar x_1)\} \vdash x_{0|1}$ are derivable in $\mathcal{IS}_{\sqsupset}$, and so is $\{x_0 \veebar x_1, x_1 \sqsupset x_0, x_0 \sqsupset x_1\} \vdash x_0$, in view of Diagonal Subsuming and Cut, in which case, by (4.11)$[x_0/(x_0 \sqsupset x_1), x_1/x_0, x_2/(x_0 \veebar x_1), x_3/(x_1 \sqsupset x_0)]$ with $\Gamma = \{x_2, x_3\}$, $\{x_0 \veebar x_1, x_1 \sqsupset x_0\} \vdash ((x_0 \sqsupset x_1) \sqsupset x_0)$ is derivable in $\mathcal{IS}_{\sqsupset}$. On the other hand, by (4.12)$[x_0/((x_0 \sqsupset x_1) \sqsupset x_0), x_1/x_0]$, (4.13) and Cut, $((x_0 \sqsupset x_1) \sqsupset x_0) \vdash x_0$ is derivable in $\mathcal{IS}^{\mathrm{PL}}_{\sqsupset}$, and so is $\{x_0 \veebar x_1, x_1 \sqsupset x_0\} \vdash x_0$, in view of Cut, in which case, by (4.11)$[x_0/(x_1 \sqsupset x_0), x_1/x_0, x_2/(x_0 \veebar x_1)]$ with $\Gamma = x_2$, (4.1) is derivable in $\mathcal{IS}^{\mathrm{PL}}_{\sqsupset}$. In this way, the "()"-non-optional "[]"-optional version of Lemma 4.1 with $j = 1$ and $\gamma = (2 \setminus 1)$ completes the argument.] □

Corollary 4.16. *Let* $\mathcal{L}$ *be a strongly* $\sqsupset$-*implicative* Σ-*logic (i.e., an axiomatic extension of* $\mathcal{I}^{\mathrm{PL}}_{\sqsupset}$*; cf. Lemma 4.14),* $\varphi \in \mathrm{Fm}_\Sigma$*,* $n \in (\omega \setminus 1)$*,* $\bar{\psi} \in \mathrm{Fm}^n_\Sigma$*,* $\bar{\phi} \in \mathrm{Fm}^*_\Sigma$*,* $v \in (\mathrm{Var} \setminus (\bigcup \mathrm{Var}[\{\varphi\} \cup ((\mathrm{img}\,\bar{\psi}) \cup (\mathrm{img}\,\bar{\phi}))]))$ *and* $\bar{\zeta} \triangleq (\sqsubset \bar{\phi})((\sqsupset v)(\bar{\psi}))$*. Then, the following hold:*

(i) $(\bar{\phi} \sqsupset ((\veebar_{\sqsupset}\bar{\psi}) \sqsupset \varphi)) \in \mathcal{L}$ *iff, for each* $i \in n$*,* $(\bar{\phi} \sqsupset (\psi_i \sqsupset \varphi)) \in \mathcal{L}$*;*
(ii) $(\bar{\phi} \sqsupset (\varphi \sqsupset (\veebar_{\sqsupset}\bar{\psi}))) \in \mathcal{L}$ *iff* $(\bar{\zeta} \sqsupset (\bar{\phi} \sqsupset (\varphi \sqsupset v))) \in \mathcal{L}$*.*

Proof. In that case, by Lemma 4.15, $\mathcal{L}$ is $\veebar_{\sqsupset}$-disjunctive. Then, Left Disjunctivity with $\Gamma = \bar{\phi}$, Right disjunctivity, (4.11), (4.12) and the induction on n immediately yield (i). Next, the "if" part of (i) with v and $\bar{\zeta} * \bar{\phi}$ instead of φ and $\bar{\phi}$, respectively, (4.11) and (4.12) yield the "only if" part of (ii). Finally, applying the substitution $[v/(\veebar_{\sqsupset}\bar{\psi})]$, the "only if" part of (i) with $\veebar_{\sqsupset}\bar{\psi}$ instead of φ, (4.11) and (4.12) imply the "if" part of (ii), as required. □

5. Disjunctive extensions of disjunctive finitely-valued logics

Lemma 5.1 (First Key Lemma). *Let* M *be a class of* $\veebar$-*disjunctive* Σ-*matrices and* $\mathcal{S} \subseteq \mathrm{Seq}_\Sigma$*. Suppose* M *is ultra-multiplicative up to isomorphisms (in particular, both it and all members of it are finite). Then, the extension* $\mathcal{L}'$ *of the logic* $\mathcal{L}$ *of* M *relatively axiomatized by* $\Re_{\veebar}(\tau_{\veebar}[\mathcal{S}])$ *is defined by* $\mathsf{M}' \triangleq (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{S}))$*.*

Proof. Let $\mathcal{A}$ be the set of all Σ-sequents true in M, in which case, by the $\veebar$-disjunctivity of members of M, $\tau_{\veebar}[\mathcal{A}] \subseteq \mathcal{A}$, and so $\tau_{\veebar}[\mathcal{A}^{\setminus 1}] = \tau_{\veebar}[\mathcal{A}]^{\setminus 1} \subseteq (\mathcal{A} \cap \mathrm{Seq}^{2\setminus 1}) = \mathcal{L} \subseteq \mathcal{L}'$, while $\mathsf{M} \subseteq \mathrm{Mod}(\mathcal{G}^{\omega}_{\veebar} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{A})$, and so, by Lemma 3.1, the sequent logic $\mathbb{S}$ of M, being deductively multiplicative, is axiomatized by $\mathcal{G}^{\omega}_{\veebar} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{A}$, for any axiom of $\mathbb{S}$ belongs to $\mathcal{A}$, and so is derivable in $\mathcal{G}^{\omega}_{\veebar} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{A}$. Then, $\mathrm{Mod}(\mathcal{L}) \supseteq \mathsf{M}$, being hereditary, includes $\mathbf{S}_*(\mathsf{M}) \supseteq \mathsf{M}'$, in which case $\mathcal{L} \subseteq \mathbb{L} \triangleq \mathcal{L}_{\mathsf{M}'}$, and so, $\mathcal{L}' \subseteq \mathbb{L}$, for $\tau_{\veebar}[\mathcal{S}]^{\setminus 1} \subseteq \mathbb{L}$, by the $\veebar$-disjunctivity of members of $\mathbf{S}_*(\mathsf{M}) \supseteq \mathsf{M}' \subseteq \mathrm{Mod}(\mathcal{S})$. Conversely, by the $\veebar$-disjunctivity of $\mathcal{L}$ and Corollary 4.3, $\mathcal{L}' \supseteq \tau_{\veebar}[\mathcal{A}^{\setminus 1}]$ is $\veebar$-disjunctive and includes $\tau_{\veebar}[\mathcal{S}]^{\setminus 1} = \tau_{\veebar}[\mathcal{S}^{\setminus 1}]$, in which case it is closed under all

Σ-substitutional instances of rules in $\mathcal{G}_{\veebar}^{2\setminus 1} \cup \mathcal{NS}_{\varnothing}^{2\setminus 1} \cup \tau_{\veebar}[(\mathcal{A} \cup \mathcal{S})^{\setminus 1}]$, and so contains all Σ-rules derivable in $\mathcal{G}_{\veebar}^{2\setminus 1} \cup \mathcal{NS}_{\varnothing}^{2\setminus 1} \cup \tau_{\veebar}[(\mathcal{A} \cup \mathcal{S})^{\setminus 1}]$ (in particular, those derivable in $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{A} \cup \mathcal{S}$, in view of Corollary 4.8). On the other hand, by Lemma 3.3, the sequent logic of $\mathsf{M}'' \triangleq (\mathbf{S}(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{S}))$ is axiomatized by $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{A} \cup \mathcal{S}$, in which case every Σ-sequent true in M'' is derivable in $\mathcal{G}_{\veebar}^{\omega} \cup \mathcal{NS}_{\varnothing} \cup \mathcal{A} \cup \mathcal{S}$, and so, in particular, $\mathcal{L}' \supseteq \mathcal{L}_{\mathsf{M}''} = \mathbb{L}$, because every Σ-rule is true in each member of $\mathsf{M}'' \setminus \mathsf{M}'$, for this is inconsistent, as required. □

Lemma 5.2. *Let $\mathcal{A}$ be a consistent $\veebar$-disjunctive Σ-matrix and $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}^{2}$. Then, the following are equivalent:*

(i) $\mathcal{A} \in \mathrm{Mod}(\mathcal{S})$;
(ii) $\mathcal{A} \in \mathrm{Mod}(\Re_{\veebar}(\mathcal{S}))$;
(iii) $\mathcal{A} \in \mathrm{Mod}(\mathcal{S}^{\setminus 1})$.

Proof. First, (i)⇒(ii) is immediate by the $\veebar$-disjunctivity of $\mathcal{A}$. Next, (ii)⇒(iii) is by the $\veebar$-disjunctivity of $\mathcal{A}$ and Corollary 4.3(ii) with $\mathcal{L} = \mathcal{L}_{\mathcal{A}}$. Finally, assume (iii) holds. Consider any $\Phi = (\Gamma \vdash \Delta) \in \mathcal{S}$, where $\Gamma, \Delta \in \mathrm{Fm}_{\Sigma}^{*}$. Then, in case $\Delta \neq \varnothing$, $\Phi \in \mathcal{S}^{\setminus 1}$, and so Φ is true in $\mathcal{A}$. Otherwise, $\Psi \triangleq (\sigma_{+1}(\Gamma) \vdash x_0) \in \mathcal{S}_{\setminus 1}$ is true in $\mathcal{A}$. Consider any $h \in \hom(\mathfrak{Tm}_{\Sigma}, \mathfrak{A})$. Take any $a \in (A \setminus D^{\mathcal{A}}) \neq \varnothing$. Let $g \in \hom(\mathfrak{Tm}_{\Sigma}, \mathfrak{A})$ extend $[x_0/a; x_{i+1}/h(x_i)]_{i \in \omega}$, in which case $\mathcal{A} \models \Psi[g]$, and so, for some $\varphi \in (\mathrm{img}\,\Gamma)$, $h(\varphi) = g(\sigma_{+1}(\varphi)) \notin D^{\mathcal{A}}$, because $g(x_0) = a \notin D^{\mathcal{A}}$. Thus, $\mathcal{A} \models \Phi[h]$, in which case Φ is true in $\mathcal{A}$, and so (i) holds, as required. □

A *([strict] Horn) universal relative* {*equality-free first-order model*} *subclass of* a class M of Σ-matrices is any subclass of M of the form $\mathsf{M} \cap \mathrm{Mod}(\mathcal{S})$, where $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}^{(2[\setminus 1])}$, in which case it is said to be *relatively axiomatized by* $\mathcal{S}$. Clearly, the intersection of any non-empty family of ([strict] Horn) universal relative subclasses of M is a ([strict] Horn) universal relative subclass of M relatively axiomatized by the union of their relative axiomatizations. Likewise, $\mathsf{M}|\varnothing$ is the strict| Horn universal relative subclass of M relatively axiomatized by $\varnothing|\{\vdash\}$, respectively. And what is more, given any $\mathcal{S}, \mathcal{T} \subseteq \mathrm{Seq}_{\Sigma}$, $((\mathsf{M} \cap \mathrm{Mod}(\mathcal{S})) \cup (\mathsf{M} \cap \mathrm{Mod}(\mathcal{T})) = (\mathsf{M} \cap \mathrm{Mod}(\{\Phi \uplus \sigma_{+m}(\Psi) \mid \Phi \in \mathcal{S}, \Psi \in \mathcal{T}, m = (\max(x_{\omega}^{-1}[\mathrm{Var}(\Phi)]) + 1)\}))$ is a universal relative subclass of M, so universal relative subclasses of M form a bounded distributive lattice. By Lemma 5.2, we also have:

Corollary 5.3. *Let M be a class of consistent $\veebar$-disjunctive Σ-matrices and $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}$. Then, the universal relative subclass of M relatively axiomatized by $\mathcal{S}$ is the Horn one relatively axiomatized by $\tau_{\veebar}[\mathcal{S}]$, and so the strict one relatively axiomatized by either $\Re_{\veebar}(\tau_{\veebar}[\mathcal{S}])$ or $\tau_{\veebar}[\mathcal{S}]^{\setminus 1}$. In particular, universal relative subclasses of M are exactly [strict] Horn ones.*

Theorem 5.4. *Let M be a class of $\veebar$-disjunctive Σ-matrices. Suppose M is ultramultiplicative up to isomorphisms (more specifically, both it and all members of it are finite). Then, the following hold:*

(i) *The mappings*

$$\begin{array}{rcl} \mathcal{L} & \mapsto & (\mathbf{S}_{*}(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{L})) \\ \mathsf{S} & \mapsto & \mathcal{L}_{\mathsf{S}} \end{array}$$

are inverse to one another dual isomorphisms between the posets of all $\veebar$-disjunctive extensions of $\mathcal{L}_{\mathsf{M}}$ and of all [{strict} Horn] universal relative subclasses of $\mathbf{S}_{*}(\mathsf{M})$, both being (finite) distributive lattices;

(ii) *for any $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}$, the universal relative subclass of $\mathbf{S}_{*}(\mathsf{M})$ relatively axiomatized by $\mathcal{S}$ corresponds to the $\veebar$-disjunctive extension of $\mathcal{L}_{\mathsf{M}}$ relatively axiomatized by $\Re_{\veebar}(\tau_{\veebar}[\mathcal{S}])$.*

(In particular, any $\veebar$-disjunctive extension of $\mathcal{L}_{\mathsf{M}}$ is finitely-relatively-axiomatizable.)

Proof. Consider any $\veebar$-disjunctive extension $\mathcal{L}$ of $\mathcal{L}_{\mathsf{M}}$. Then, by the second assertion of the "[]"-non-optional version of Lemma 4.1, $\mathcal{L}$ is $\veebar$-multiplicative, in which case it includes $\Re_{\veebar}(\mathcal{L})$, and so the extension $\mathcal{L}'$ of $\mathcal{L}$ relatively axiomatized by $\Re_{\veebar}(\mathcal{L})$. Conversely, by Corollary 4.3(ii), $\mathcal{L}'$ includes $\mathcal{L}^{2\setminus 1} = \mathcal{L}$, in which case $\mathcal{L} = \mathcal{L}'$, and so, by Lemma 5.1, $\mathcal{L}$ is defined by the strict Horn universal relative subclass $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{L})$ of $\mathbf{S}_*(\mathsf{M})$. In this way, Corollaries 4.3(i), 5.3 and Lemma 5.1 complete the argument. □

5.1. **Implicative case.** Let $\theta_{\sqsupset} : \mathrm{Seq}_{\Sigma}^{2\setminus 1} \to \mathrm{Fm}_{\Sigma}, (\Gamma \vdash \varphi) \mapsto (\varepsilon(\Gamma) \sqsupset \varphi)$. Then, the strict Horn universal relative subclass of any class M of $\sqsupset$-implicative Σ-matrices relatively axiomatized by any $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}^{2\setminus 1}$ is relatively axiomatized by $\theta_{\sqsupset}[\mathcal{S}] = \Re_{\veebar}(\theta_{\sqsupset}[\mathcal{S}])$. In this way, by Corollaries 4.3, 5.3, Lemma 4.14 and Theorem 5.4, we immediately get:

Corollary 5.5. *Let M be a class of $\sqsupset$-implicative $\veebar$-disjunctive Σ-matrices. Suppose M is ultra-multiplicative up to isomorphisms (more specifically, both it and all members of it are finite). Then, the following hold:*

(i) *The mappings*

$$\begin{array}{rcl} \mathcal{L} & \mapsto & (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{L})) \\ \mathsf{S} & \mapsto & \mathcal{L}_{\mathsf{S}} \end{array}$$

are inverse to one another dual isomorphisms between the posets of all axiomatic extensions of $\mathcal{L}_{\mathsf{M}}$ and of all [{strict} Horn] universal relative subclasses of $\mathbf{S}_*(\mathsf{M})$, both being (finite) distributive lattices;

(ii) *for any $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}$, the universal relative subclass of $\mathbf{S}_*(\mathsf{M})$ relatively axiomatized by $\mathcal{S}$ corresponds to the axiomatic extension of $\mathcal{L}_{\mathsf{M}}$ relatively axiomatized by $\theta_{\sqsupset}[\tau_{\veebar}[\mathcal{S}]^{\setminus 1}]$;*

(iii) *$\veebar$-disjunctive extensions of $\mathcal{L}_{\mathsf{M}}$ are exactly $\sqsupset$-implicative/axiomatic ones.*

(In particular, any axiomatic extension of $\mathcal{L}_{\mathsf{M}}$ is relatively axiomatized by a finite axiomatic Σ-calculus.)

5.2. **The unitary finitely-valued case with equality determinant.**

Lemma 5.6. *Let $\mathcal{A}$ be a finite Σ-matrix with finite equality determinant $\Im$ and $\mathsf{K} \subseteq \mathbf{S}(\mathcal{A})$. Then, relative universal subclasses of K are exactly relatively hereditary subclasses of it.*

Proof. Submatrices of $\mathcal{A}$ are uniquely determined by (and so identified with) the carriers of their underlying algebras. And what is more, any relatively hereditary subclass S of K is the union of the finite set $\{\mathsf{K} \cap \mathbf{S}(\mathcal{B}) \mid \mathcal{B} \in \max(\mathsf{S})\}$, for K is finite, because $\mathcal{A}$ is so. Consider any $\mathcal{B} \in \max(\mathsf{S})$ and any $\mathcal{C} \in (\mathsf{K} \setminus \mathbf{S}(\mathcal{B}))$, in which case $C \nsubseteq B$, and so there is some $c \in (C \setminus B) \neq \varnothing$. Let $\Phi^{0/1}_{\mathcal{C}[,c]} \triangleq (\Im^{\mathcal{A}}_{c,+/-} \cap \bigcup_{b\in B} \Im^{\mathcal{A}}_{b,-/+})$, in which case $\Phi_{\mathcal{C},c} \in \mathrm{Seq}_{\Sigma}$ is not true in $\mathcal{C}$ under $[x_0/c]$ but is true in $\mathcal{B}$ (in particular, in $\mathsf{K} \cap \mathbf{S}(\mathcal{B})$), because every $b \in B$ is distinct from $c \notin B$, in which case, as $\Im$ is an equality determinant for $\mathcal{A}$, there is some $\iota \in \Im$ such that either $\iota^{\mathfrak{A}}(c) \in D^{\mathcal{A}} \not\ni \iota^{\mathfrak{A}}(b)$, and so $\iota \in \Phi^0_{\mathcal{C},c}$, or $\iota^{\mathfrak{A}}(c) \notin D^{\mathcal{A}} \ni \iota^{\mathfrak{A}}(b)$, and so $\iota \in \Phi^1_{\mathcal{C},c}$ (in particular, in any case, $\mathcal{B} \models \Phi_{\mathcal{C},c}[x_0/b]$). Thus, $\mathsf{K} \cap \mathbf{S}(\mathcal{B})$ is the universal relative subclass of K relatively axiomatized by $\{\Phi_{\mathcal{C}} \mid \mathcal{C} \in (\mathsf{K} \setminus \mathbf{S}(\mathcal{B}))\}$, as required. □

It is remarkable that the proof of Lemma 5.6, being constructive, provides an effective (though non-deterministic, because of choice of some $c \in (C \setminus B)$) procedure of finding finite relative axiomatizations of relatively hereditary subclasses of classes of submatrices of finite matrices with equality determinant. Then, combining Theorem 5.4 with Lemma 5.6, we eventually get:

Corollary 5.7. *Let $\mathcal{A}$ be a $\veebar$-disjunctive Σ-matrix with equality determinant. Then, the following hold:*

(i) *The mappings*

$$\begin{aligned} \mathcal{L} &\mapsto (\mathbf{S}_*(\mathcal{A}) \cap \mathrm{Mod}(\mathcal{L})) \\ \mathsf{S} &\mapsto \mathcal{L}_{\mathsf{S}} \end{aligned}$$

are inverse to one another dual isomorphisms between the posets of all $\veebar$-disjunctive extensions of $\mathcal{L}_{\mathcal{A}}$ and of all relatively hereditary subclasses of $\mathbf{S}_*(\mathcal{A})$, both being finite distributive lattices;

(ii) *for any $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}$, the relatively hereditary subclass of $\mathbf{S}_*(\mathcal{A})$ relatively axiomatized by $\mathcal{S}$ corresponds to the $\veebar$-disjunctive extension of $\mathcal{L}_{\mathcal{A}}$ relatively axiomatized by $\Re_{\veebar}(\tau_{\veebar}[\mathcal{S}])$;*

(iii) *for any $\mathsf{K} \subseteq \mathbf{S}_*(\mathcal{A})$, the $\veebar$-disjunctive extension of $\mathcal{L}_{\mathcal{A}}$ defined by K corresponds to $\mathbf{S}_*(\mathsf{K})$.*

In particular, any $\veebar$-disjunctive extension of $\mathcal{L}_{\mathcal{A}}$ is finitely-relatively-axiomatizable.

As a matter of fact, despite of the alternative appearing in the formulation of Corollary 5.3, $\Re_{\veebar}((\tau_{\veebar}[)\mathcal{S}(]))$ cannot be replaced by $(\tau_{\veebar}[)\mathcal{S}(])^{\backslash 1}$ in the formulation(s) of Lemma 5.1 (resp., Corollaries 4.3, 5.7 and Theorem 5.4), as we show in Subsubsection 7.2.4 below.

5.2.1. *Implicative case.* Likewise, combining Corollary 5.5 with Lemma 5.6, we also get:

Corollary 5.8 (Theorem 3.5 of [24])**.** *Let $\mathcal{A}$ be an $\sqsupset$-implicative $\veebar$-disjunctive Σ-matrix with equality determinant. Then, the following hold:*

(i) *The mappings*

$$\begin{aligned} \mathcal{L} &\mapsto (\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{L})) \\ \mathsf{S} &\mapsto \mathcal{L}_{\mathsf{S}} \end{aligned}$$

are inverse to one another dual isomorphisms between the posets of all axiomatic extensions of $\mathcal{L}_{\mathsf{M}}$ and of all relatively hereditary subclasses of $\mathbf{S}_*(\mathsf{M})$, both being finite distributive lattices;

(ii) *for any $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}$, the relatively hereditary subclass of $\mathbf{S}_*(\mathsf{M})$ relatively axiomatized by $\mathcal{S}$ corresponds to the axiomatic extension of $\mathcal{L}_{\mathsf{M}}$ relatively axiomatized by $\theta_{\sqsupset}[\tau_{\veebar}[\mathcal{S}]^{\backslash 1}]$;*

(iii) *$\veebar$-disjunctive extensions of $\mathcal{L}_{\mathsf{M}}$ are exactly $\sqsupset$-implicative/axiomatic ones;*

(iv) *for any $\mathsf{K} \subseteq \mathbf{S}_*(\mathcal{A})$, the axiomatic extension of $\mathcal{L}_{\mathcal{A}}$ corresponding to $\mathbf{S}_*(\mathsf{K})$ is defined by K.*

In particular, any axiomatic extension of $\mathcal{L}_{\mathsf{M}}$ is relatively axiomatized by a finite axiomatic Σ-calculus.

6. Finite Hilbert-style axiomatizations

Let $\mathcal{A}$ be a finite Σ-matrix with finite equality determinant $\Im \ni x_0$. Then, elements of $\Im[\Sigma(\lceil n)]$ (where $n \in \omega$) are referred to as *$\Im$-compound connectives of Σ (of arity n)* — these are secondary (n-ary) connectives of Σ.

According to [19],[2] a Σ-*sequent(ial) $\Im$-table (of rank $(0,0)$) for $\mathcal{A}$* is any couple $\mathcal{T} = \langle \lambda_{\mathcal{T}}, \rho_{\mathcal{T}} \rangle$ of mappings from $\Im[\Sigma \setminus (\Sigma\lceil 0)] \setminus \Im$ to $\wp_{\omega}(\wp_{\omega}(\Im[\mathrm{Var}_{\omega}])^2)$ such that,

[2]Although, as opposed to the present study, the mentioned one deals with sequent sides as finite rather sequences than sets, its notions and results, being properly re-formulated, are clearly retained within the formalism adopted here.

for every $F \in \Sigma$ of arity $n \in (\omega \setminus 1)$ and all $\iota \in \Im$ with $\iota(F) \notin \Im$, each element of $(\lambda|\rho)_{\mathcal{T}}(\iota(F)) \subseteq \wp_\omega(\Im[\mathrm{Var}_n])^2$ is disjoint, while

$$\mathcal{A} \models \forall \bar{x}_n(((0|1) : \iota(F)) \leftrightarrow (\bigwedge(\lambda|\rho)_{\mathcal{T}}(\iota(F)))), \tag{6.1}$$

in which case all elements of $(\boldsymbol{\lambda}|\boldsymbol{\rho})_{\mathcal{T}}(\iota(F)) \triangleq (\uplus(0|1) : \iota(F)))[(\rho|\lambda)_{\mathcal{T}}(\iota(F))]$ are true in $\mathcal{A}$, while, according to (the constructive proof of) Theorem 1 therein, it exists (and can be found effectively, in case Σ is finite), and so, from now on, unless otherwise specified, we fix any one.

Let $\mathcal{A}' \triangleq \{(i : \iota(c)) \mid i \in 2, \iota \in \Im, c \in (\Sigma{\restriction}0), \mathcal{A} \models (i : \iota(c))\}$.

Next, the set $\mathrm{Ax}(\Im)$ of all disjoint elements of $\wp_\omega(\Im)^2$ is finite and *partially* ordered by $\preceq_{[\eth]}$, because, for all $\phi, \psi \in \mathrm{Tm}^1_\Sigma$, $\phi = x_0 = \psi$, whenever $\phi(\psi) = x_0$. Let $\mathrm{Ax}(\mathcal{A}) \triangleq \{\Phi \in \mathrm{Ax}(\Im) \mid \mathcal{A} \models \forall x_0 \Phi\}$ and $\mathcal{A}''_{[\eth]} \triangleq \min_{\preceq_{[\eth]}}(\mathrm{Ax}(\mathcal{A}))$.

6.1. **Disjunctive case.** Here, $\mathcal{A}$ is supposed to be $\veebar$-disjunctive, in which case we have:

Remark 6.1. When $\veebar$ is a primary binary connective of Σ (in particular, $\veebar \notin \Im$), one can always take $\lambda_{\mathcal{T}}(\veebar) = \{x_0 \vdash, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\veebar) = \{\vdash \{x_0, x_1\}\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\veebar) = \{(x_0 \veebar x_1) \vdash \{x_0, x_1\}\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\veebar) = \{x_0 \vdash (x_0 \veebar x_1), x_1 \vdash (x_0 \veebar x_1)\}$, and so their elements are all derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing$. $\square$

Let $\mathcal{A}'''_{[\not\varsigma]} \triangleq (\bigcup\{\boldsymbol{\lambda}_{\mathcal{T}}(\iota(F)) \cup \boldsymbol{\rho}_{\mathcal{T}}(\iota(F)) \mid (\mathrm{Var}_1 \times \{\veebar[,\varsigma]\}) \not\ni \langle \iota, F\rangle \in (\Im \times (\Sigma \setminus (\Sigma{\restriction}0))), \iota(F) \notin \Im)\})$ [where $\varsigma \in (\Sigma \setminus (\Sigma{\restriction}0))$] and $\mathcal{A}_{(\eth)[(,)\not\varsigma]} \triangleq (\mathcal{A}' \cup \mathcal{A}''_{(\eth)} \cup \mathcal{A}'''_{[\not\varsigma]})$, in which case this is finite, whenever Σ is so, while every element of it is true in $\mathcal{A}$.

Lemma 6.2 (Second Key Lemma)**.** *Any Σ-sequent true in $\mathcal{A}$ is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$.*

Proof. By Theorem 2 of [19], because each rule of the sequent Σ-calculus $\mathcal{S}^{(0,0)}_{\mathcal{A},\mathcal{T}}$ specified in Definition 1 therein is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, as we argue throughout the rest of the proof.

First, for every axiom Φ in the item (i) of Definition 1 of [19], there is some $\sigma \in \mathrm{Sb}_\Sigma$ such that $\Psi \triangleq \sigma(x_0 \vdash x_0) \preceq_\eth \Phi$, in which case Ψ, being a Σ-substitutional instance of Reflexivity, is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, and so is Φ, in view of Diagonal Subsuming. Likewise, for every axiom Φ in the items (iii,iv) of Definition 1 of [19], there is some $\Psi \in \mathcal{A}'$ such that $\Psi \preceq_\eth \Phi$, in which case Ψ is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, and so is Φ, in view of Diagonal Subsuming. Next, for every axiom Φ in the item (ii) of Definition 1 of [19], there is some $\Psi \in \mathrm{Ax}(\mathcal{A})$ such that $\Psi \preceq \Phi$, in which case there is some $\Upsilon \in \mathcal{A}''_{[\eth]}$ such that $\Upsilon \preceq_{[\eth]} \Psi$, and so $\Upsilon \preceq \Phi$. Then, there is some $\sigma \in \mathrm{Sb}_\Sigma$ such that $\Omega \triangleq \sigma(\Upsilon) \preceq_\eth \Phi$, in which case Ω, being a Σ-substitutional instance of $\Upsilon \in \mathcal{A}''_{[\eth]} \subseteq \mathcal{A}_{[\eth]}$, is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, and so is Φ, in view of Diagonal Subsuming.

Finally, consider any $F \in (\Sigma \setminus (\Sigma{\restriction}0))$ and any $\iota \in \Im$ such that $\iota(F) \notin \Im$. We start from proving that

$$\frac{\lambda_{\mathcal{T}}(\iota(F)) \cup \rho_{\mathcal{T}}(\iota(F))}{\vdash} \tag{6.2}$$

is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$. For note that, by (6.1), (6.2) is true in $\mathcal{A}$, and so is every element of $S \triangleq (\overline{\Omega} \rhd (\vdash))) \subseteq \wp_\omega(\Im[\mathrm{Var}_\omega])^2$, where $\overline{\Omega}$ is any enumeration of $(\lambda_{\mathcal{T}}(\iota(F)) \cup \rho_{\mathcal{T}}(\iota(F)))$. Consider any $\Phi = (\Gamma \vdash \Delta) \in S$. If it is not disjoint, then there is some $\sigma \in \mathrm{Sb}_\Sigma$ such that $\Psi \triangleq \sigma(x_0 \vdash x_0) \preceq_\eth \Phi$, in which case Ψ, being a Σ-substitutional instance of Reflexivity, is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, and so is Φ, in view of Diagonal Subsuming. Otherwise, for each $i \in \omega$, $(\Phi{\restriction}i) \triangleq (\{\iota \in \Im \mid \iota(x_i) \in \Gamma\} \vdash \{\iota \in \Im \mid \iota(x_i) \in \Delta\})) \in \mathrm{Ax}(\Im)$, in which case $((\Phi{\restriction}i)[x_0/x_i]) \preceq_\eth \Phi$, and so $(\Phi\restriction i) \preceq \Phi$, while $(\Gamma|\Delta) = (\bigcup_{i\in\omega} \pi_{0|1}(\Phi\restriction i))$, and so, if, for each $i \in \omega$, $(\Phi\restriction i)$ was not true in $\mathcal{A}$, then there would be some $a_i \in A$ such that $\phi^{\mathfrak{A}}[x_0/a_i]$ would /not be in $D^{\mathcal{A}}$, for every $\phi \in \pi_{0/1}(\Phi\restriction i)$ (in particular, Φ would not be true in $\mathcal{A}$ under $\bar{a} \circ x_\omega^{-1}$). Hence, $(\Phi\restriction i) \in \mathrm{Ax}(\mathcal{A})$, for some $i \in \omega$, in which case there is some $\Upsilon \in \mathcal{A}''_{[\eth]}$ such that $\Upsilon \preceq_{[\eth]} (\Phi\restriction i)$, and so $\Upsilon \preceq \Phi$. In this way, there is some $\sigma \in \mathrm{Sb}_\Sigma$ such that $\Xi \triangleq \sigma(\Upsilon) \preceq_{\eth} \Phi$, in which case Ξ, being a Σ-substitutional instance of $\Upsilon \in \mathcal{A}''_{[\eth]} \subseteq \mathcal{A}_{[\eth]}$, is derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, and so is Φ, in view of Diagonal Subsuming. Thus, in any case, Φ is derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$. Therefore, applying $|\lambda_{\mathcal{T}}(\iota(F)) \cup \rho_{\mathcal{T}}(\iota(F))|$ times Lemma 3.1, we eventually conclude that (6.2) is derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$. On the other hand, this is clearly multiplicative, and so deductively so, in view of Lemma 3.2. Hence, $(\uplus((0|1) : \iota(F)))(6.2)$ is derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$. In this way, as every element of $(\boldsymbol{\lambda}|\boldsymbol{\rho})_{\mathcal{T}}(\iota(F))$, being either in $\mathcal{A}''' \subseteq \mathcal{A}_{[\eth]}$ (in particular, derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$), if $\langle \iota, F\rangle \neq \langle x_0, \underline{\vee}\rangle$, or derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, otherwise, in view of Remark 6.1, is derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, we conclude that $\dfrac{(\lambda|\rho)_{\mathcal{T}}(\iota(F))}{(0|1) : \iota(F))}$ is derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, and so is each Σ-substitutional instance of it, in which case, by the deductive multiplicativity of $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, each rule in the item (v) of Definition 1 of [19] is derivable in $\mathcal{G}^\omega_{\underline{\vee}} \cup \mathcal{NS}^\omega_\varnothing \cup \mathcal{A}_{[\eth]}$, as required. □

In this way, combining Lemma 6.2 with Corollary 4.8, we eventually get:

Corollary 6.3. *Any [purely-]single-conclusion Σ-sequent true in $\mathcal{A}$ is derivable in $\mathcal{G}^{2[\backslash 1]}_{\underline{\vee}} \cup \mathcal{NS}^{2[\backslash 1]}_\varnothing \cup \tau_{\underline{\vee}}[\mathcal{A}^{[\backslash 1]}_{(\eth)}]$.*

Theorem 6.4. *The logic of $\mathcal{A}$ is axiomatized by $\mathcal{D}_{(\eth)} \triangleq (\mathcal{B}_{\underline{\vee}} \cup \Re_{\underline{\vee}}(\tau_{\underline{\vee}}[\mathcal{A}_{(\eth)}]))$.*

Proof. First, in view of the $\underline{\vee}$-disjunctivity of $\mathcal{A}$ and Theorem 4.12, elements of $\mathcal{D}_{(\eth)}$ are true in $\mathcal{A}$, for those of $\tau_{\underline{\vee}}[\mathcal{A}_{(\eth)}]$ are so, because those of $\mathcal{A}_{(\eth)}$ are so.

Conversely, by Corollary 4.3 and Theorem 4.12, $\mathcal{L} \triangleq \mathcal{L}_{\mathcal{D}_{(\eth)}}$ is $\underline{\vee}$-disjunctive and includes $\tau_{\underline{\vee}}[\mathcal{A}_{(\eth)}]^{2\backslash 1} = \tau_{\underline{\vee}}[\mathcal{A}^{\backslash 1}_{(\eth)}]$, in which case it is closed under every Σ-substitutional instance of each element of $\mathcal{G}^{2\backslash 1}_{\underline{\vee}} \cup \mathcal{NS}^{2\backslash 1}_\varnothing \cup \tau_{\underline{\vee}}[\mathcal{A}^{\backslash 1}_{(\eth)}]$, and so contains all Σ-rules derivable in $\mathcal{G}^{2\backslash 1}_{\underline{\vee}} \cup \mathcal{NS}^{2\backslash 1}_\varnothing \cup \tau_{\underline{\vee}}[\mathcal{A}^{\backslash 1}_{(\eth)}]$ [including all Σ-rules true in $\mathcal{A}$, in view of Corollary 6.3]. □

6.1.1. *Implicative subcase.* Here, it is supposed that $\mathcal{A}$ is $\sqsupset$-implicative, in which case it is $\underline{\vee}$-disjunctive, where $\underline{\vee} \triangleq \underline{\vee}_{\sqsupset} \notin \Sigma$, and so $((\mathrm{Var}_1 \times \{\underline{\vee}\}) \cap (\Im \times (\Sigma \backslash (\Sigma\restriction 0)))) = \varnothing$.

Remark 6.5. When $\sqsupset$ is a primary binary connective of Σ (in particular, $\sqsupset \notin \Im$), one can always take $\lambda_{\mathcal{T}}(\sqsupset) = \{\vdash x_0, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sqsupset) = \{x_0 \vdash x_1\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sqsupset)_{[\backslash 1]} = \{(4.12)\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\underline{\vee})_{[\backslash 1]} = \{\vdash \{x_0, (x_0 \sqsupset x_1)\}, x_1 \vdash (x_0 \sqsupset x_1)\}$, and so elements of both $\theta_{\sqsupset}[\tau_{\underline{\vee}}[\boldsymbol{\lambda}_{\mathcal{T}}(\sqsupset)_{\backslash 1}]] = \{\theta_{\sqsupset}(4.12)\}$ and $\theta_{\sqsupset}[\tau_{\underline{\vee}}[\boldsymbol{\rho}_{\mathcal{T}}(\sqsupset)_{\backslash 1}]] = \{((4.16), (4.14)[x_0/x_1, x_1/x_0]\}$ are derivable in $\mathcal{I}_{\sqsupset}$, in view of Lemma 4.14, (4.11), (4.12), (4.14) and (4.16). □

Theorem 6.6. *The logic of $\mathcal{A}$ is axiomatized by $\mathfrak{J}_{(\eth)[(,)\not\sqsupset]} \triangleq (\mathcal{I}^{\mathrm{PL}}_{\sqsupset} \cup \theta_{\sqsupset}[\tau_{\underline{\vee}}[\mathcal{A}^{\backslash 1}_{(\eth)[(,)\not\sqsupset]}]])$.*

Proof. First, by Remark 6.5, we have $\mathcal{L} \triangleq \mathcal{L}_{\mathfrak{J}_{(\eth)}} = \mathcal{L}_{\mathfrak{J}_{(\eth,)\not\sqsupset}}$. Next, in view of the $\sqsupset$-implicativity (in particular, $\underline{\vee}$-disjunctivity) of $\mathcal{A}$, by Lemma 4.14, all elements of $\mathfrak{J}_{(\eth)}$ are true in $\mathcal{A}$, for those of $\theta_\theta[\tau_{\underline{\vee}}[\mathcal{A}^{\backslash 1}_{(\eth)}]]$ are so, because those of $\tau_{\underline{\vee}}[\mathcal{A}^{\backslash 1}_{(\eth)}]$ are so, as those of $\mathcal{A}^{\backslash 1}_{(\eth)}$ are so, since those of $\mathcal{A}_{(\eth)}$ are so. Conversely, by (4.12), each $\mathcal{R} \in \tau_{\veebar}[\mathcal{A}^{\setminus 1}_{(\eth)}]$ belongs to $\mathcal{L}$, for $\theta_{\sqsupset}(\mathcal{R}) \in \mathcal{J}_{(\eth)} \subseteq \mathcal{L}$, and so does every Σ-substitutional instance of it. And what is more, by Lemma 4.15, $\mathcal{L}$ is $\veebar$-disjunctive, in which case it is closed under every Σ-substitutional instance of each element of $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \mathcal{NS}^{2\setminus 1}_{\varnothing} \cup \tau_{\veebar}[\mathcal{A}^{\setminus 1}_{(\eth)}]$, and so contains all Σ-rules derivable in $\mathcal{G}^{2\setminus 1}_{\veebar} \cup \mathcal{NS}^{2\setminus 1}_{\varnothing} \cup \tau_{\veebar}[\mathcal{A}^{\setminus 1}_{(\eth)}]$ {including all Σ-rules true in $\mathcal{A}$, in view of Corollary 6.3}. $\square$

Since $\vdash$ is not true in any Σ-matrix (in particular, in $\mathcal{A}$), it does not belong to $\mathcal{A}_{[\eth]}$, for every element of this is true in $\mathcal{A}$. Therefore, combining Corollary 4.16 with Theorem 6.6, we eventually get:

Corollary 6.7. *The logic of $\mathcal{A}$ is axiomatized by* $\mathcal{K}_{(\eth)[(,)\not\Delta]} \triangleq (\mathcal{I}^{\mathrm{PL}}_{\sqsupset} \cup \{\veebar\varepsilon(\Delta) \mid \Delta \in \wp_{\omega\setminus 1}(\mathrm{Fm}_\Sigma), (\vdash \Delta) \in \mathcal{A}_{(\eth)[(,)\not\Delta]}\} \cup \theta_{\sqsupset}[(\mathcal{A}^{\setminus 1}_{(\eth)[(,)\not\Delta]} \cap \mathrm{Seq}^2_\Sigma) \cup \{((\sqsubset \varepsilon(\Gamma))[(\sqsupset x_0)[\Delta]] \cup \Gamma \cup \{\varphi\}) \vdash x_0 \mid \varphi \in \mathrm{Fm}_\Sigma, \Gamma \in \wp_\omega(\mathrm{Fm}_\Sigma), \Delta \in \wp_{\omega\setminus 1}(\mathrm{Fm}_\Sigma), ((\Gamma \cup \{\varphi\}) \vdash \Delta) \in \sigma_{+1}[\mathcal{A}^{\setminus 1}_{(\eth)[(,)\not\Delta]}]\}])$.

7. Application and examples

Here, we follow Sections 5, 6 and use Corollary 6.7/"4.13 and Theorem 6.4" as well as Corollary 5.8/5.7 tacitly in the implicative/disjunctive case, respectively.

7.1. Disjunctive and implicative positive fragments of the classical logic.

Here, we deal with the signature $\Sigma^{(\supset)}_{+[,01]} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}](\cup\{\supset\}))$. By $\mathfrak{D}^{(\supset)}_{n[,01]}$, where $n(= 2) \in (\omega \setminus 1)$, we denote the $\Sigma^{(\supset)}_{+[,01]}$-algebra such that $\mathfrak{D}^{(\supset)}_{n[,01]} \restriction \Sigma_{+[,01]}$ is the [bounded] distributive lattice given by the chain poset $n \subseteq \wp(\omega)$ (and $(i \supset^{\mathfrak{D}^{\supset}_{2[,01]}} j) \triangleq (\max(1 - i, j)$, for all $i, j \in 2$). Then, the logic of the $\vee$-disjunctive (and $\supset$-implicative) $\mathcal{D}^{(\supset)}_{2[,01]} \triangleq \langle \mathfrak{D}^{(\supset)}_{2[,01]}, \{1\}\rangle$ with equality determinant $\Im = \{x_0\}$ {cf. Example 1 of [19]} is the $\Sigma^{(\supset)}_{+[,01]}$-fragment of the classical logic. Throughout the rest of this subsection, it is supposed that $\Sigma \subseteq \Sigma^{(\supset)}_{+,01}$ and $\mathcal{A} = (\mathcal{D}^{(\supset)}_{2,01} \restriction \Sigma)$, in which case $\mathcal{A}''_{\{\eth\}} = \varnothing$.

First, in case $\Sigma = \{\supset\}$, both $\mathcal{A}'''_{\not\supset}$ and $\mathcal{A}'$ are empty, and so is $\mathcal{A}_{\{\eth,\}\not\supset}$. In this way, we have the following well-known result:

Corollary 7.1. *The $\{\supset\}$-fragment of the classical logic is axiomatized by $\mathcal{I}^{\mathrm{PL}}_{\supset}$. In particular, the latter can be replaced by any other Hilbert-style axiomatization of the former in the formulations of Theorem 6.6 and Corollary 6.7.*

Likewise, in case $\Sigma = \{\vee\}$, both $\mathcal{A}'$ and $\mathcal{A}'''$ are empty, and so is $\mathcal{A}_{\{\eth\}}$. In this way, we get the following seemingly new result:

Corollary 7.2. *The $\{\vee\}$-fragment of the classical logic is axiomatized by $\mathcal{B}_\vee$. In particular, the latter can be replaced by any other Hilbert-style axiomatization of the former in the formulation of Theorem 6.4.*

Next, let $\Sigma = \Sigma_+$. Then, $\mathcal{A}' = \varnothing$, while one can take $\lambda_{\mathcal{T}}(\wedge) = \{\{x_0, x_1\} \vdash\}$ and $\rho_{\mathcal{T}}(\wedge) = \{\vdash x_0, \vdash x_1\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\wedge) = \{(x_0 \wedge x_1) \vdash x_0, (x_0 \wedge x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\wedge) = \{\{x_0, x_1\} \vdash (x_0 \wedge x_1)\}$, and so $\mathcal{A}_{\{\eth\}} = \mathcal{A}''' = \{(x_0 \wedge x_1) \vdash x_0, (x_0 \wedge x_1) \vdash x_1, \{x_0, x_1\} \vdash (x_0 \wedge x_1)\}$. Thus, we get:

Corollary 7.3. *The Σ_+-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_+$ resulted from $\mathcal{B}_\vee$ by adding the following rules:*

$$C_1 \quad \frac{(x_1 \wedge x_2) \vee x_0}{x_1 \vee x_0} \qquad C_2 \quad \frac{(x_1 \wedge x_2) \vee x_0}{x_2 \vee x_0} \qquad C_3 \quad \frac{\{x_1 \vee x_0, x_2 \vee x_0\}}{(x_1 \wedge x_2) \vee x_0}$$

It is remarkable that the calculus $\mathcal{PC}_+$ consists of seven rules, while that which was found in [4] has nine rules. This demonstrates the practical applicability of our generic approach (more precisely, its factual ability to result in really "good" calculi to be enhanced a bit more by replacing appropriate pairs of rules/premises with single ones upon the basis of Corollary 4.13 and rules C_i, where $i \in (4 \setminus 1)$, whenever it is possible, to be done below tacitly — "on the fly").

Likewise, let $\Sigma = \Sigma_+^{\supset}$. Then, $\mathcal{A}' = \varnothing$, and so, taking Remark 6.1 into account, we have the following well-known result:

Corollary 7.4. *The $\Sigma_+^{\supset}$-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_+^{\supset}$ resulted from $\mathcal{I}_{\supset}^{\mathrm{PL}}$ by adding the following axioms:*

$$(x_0 \wedge x_1) \supset x_i \qquad\qquad x_0 \supset (x_1 \supset (x_0 \wedge x_1))$$
$$x_i \supset (x_0 \vee x_1) \qquad (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \vee x_1) \supset x_2))$$

where $i \in 2$.

Finally, let $\Sigma = \Sigma_{+,01}^{[\supset]}$, in which case $\mathcal{A}'''_{\{\eth\}}$ is as above, while $\mathcal{A}' = \{\vdash \top, \bot \vdash\}$, and so we get:

Corollary 7.5. *The $\Sigma_{+,01}^{[\supset]}$-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_{+,01}^{[\supset]}$ resulted from $\mathcal{PC}_+^{[\supset]}$ by adding the axiom $\top$ and the rule $\dfrac{\bot \vee x_0}{x_0}$ [resp., the axiom $\bot \supset x_0$].*

7.2. **Miscellaneous four-valued expansions of Dunn-Belnap's four-valued logic.** Let $\Sigma_{\sim,+[,01]}^{(\supset)} \triangleq (\Sigma_{+[,01]}^{(\supset)} \cup \{\sim\})$, where $\sim$ — weak negation — is unary. Here, it is supposed that $\Sigma \supseteq \Sigma_{\sim,+[,01]}$, $(\mathfrak{A}{\restriction}\Sigma_{\sim,+[,01]}) = \mathfrak{DM}_{4[,01]}$, where $(\mathfrak{DM}_{4[,01]}{\restriction}\Sigma_{+[,01]}) \triangleq \mathfrak{D}^2_{2[,01]}$, while $\sim^{\mathfrak{DM}_{4[,01]}}\langle i,j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i, j \in 2$, in which case we use the following standard notations for elements of 2^2 going back to [2]:

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle,$$

and $\mathcal{A} \triangleq \langle \mathfrak{A}, \{\mathsf{b},\mathsf{t}\}\rangle$, in which case it is $\vee$-disjunctive, while $\Im = \{x_0, \sim x_0\}$ is an equality determinant for it {cf. Example 2 of [19]}, whereas $\mathcal{A}''_{\langle\eth\rangle} = \varnothing$. Since the logic $DB_{4[,01]}$ of $\mathcal{DM}_{4[,01]} \triangleq (\mathcal{A}{\restriction}\Sigma_{\sim,+[,01]})$ is the [bounded version of] Dunn-Belnap's logic [2, 3], the logic of $\mathcal{A}$ is a four-valued expansion of $DB_{4[,01]}$.

First, let $\Sigma = \Sigma_{\sim,+}$, in which case $\mathcal{A}' = \varnothing$, while the case of the $\Im$-compound connective $\wedge$ is as in the previous subsection, for $x_0 \in \Im$, whereas others not belonging to $\Im$ (i.e., distinct from $\sim$) but $\vee$ are as follows. First of all, one can take $\lambda_{\mathcal{T}}(\sim\vee) = \{\{\sim x_0, \sim x_1\} \vdash\}$ and $\rho_{\mathcal{T}}(\sim\vee) = \{\vdash \sim x_0, \vdash \sim x_1\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim\vee) = \{\sim(x_0 \vee x_1) \vdash \sim x_0, \sim(x_0 \vee x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim\vee) = \{\{\sim x_0, \sim x_1\} \vdash \sim(x_0 \vee x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim\wedge) = \{\sim x_0 \vdash, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim\wedge) = \{\vdash \{\sim x_0, \sim x_1\}\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim\wedge) = \{\sim(x_0 \wedge x_1) \vdash \{\sim x_0, \sim x_1\}\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim\wedge) = \{\sim x_0 \vdash \sim(x_0 \wedge x_1), \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Finally, one can take $\lambda_{\mathcal{T}}(\sim\sim) = \{x_0 \vdash\}$ and $\rho_{\mathcal{T}}(\sim\sim) = \{\vdash x_0\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim\sim) = \{\sim\sim x_0 \vdash x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim\sim) = \{x_0 \vdash \sim\sim x_0\}$. In this way, we get:

Corollary 7.6. *DB_4 is axiomatized by the calculus $\mathcal{D}$ resulted from $\mathcal{PC}_+$ by adding the following rules:*

$$\begin{array}{ccc} NN & ND & NC \\ \dfrac{x_1 \vee x_0}{\sim\sim x_1 \vee x_0}\updownarrow & \dfrac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \vee x_2) \vee x_0}\updownarrow & \dfrac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \wedge x_2) \vee x_0}\updownarrow \end{array}$$

The calculus $\mathcal{D}$ has 13 rules, while the very first axiomatization of DB_4 discovered in [15] (cf. Definition 5.1 and Theorem 5.2 therein) has 15 rules, "two rules win" being just due to the advance of the present work with regard to [4] (cf. the previous subsection).

Now, let $\Sigma = \Sigma_{\sim,+,01}$, in which case $\mathcal{A}'''$ is as above, while $\mathcal{A}' = \{\top, \sim\bot, \bot \vdash, \sim\top \vdash\}$, and so we get:

Corollary 7.7. *$DB_{4,01}$ is axiomatized by the calculus $\mathcal{D}_{01}$ resulted from $\mathcal{D} \cup \mathcal{PC}_{+,01}$ by adding the axiom $\sim\bot$ and the rule $\frac{\sim\top \vee x_0}{x_0}$.*

7.2.1. *The classically-negative expansion.* Let $\Sigma^{(\supset)}_{\simeq,+[,01]} \triangleq (\Sigma^{(\supset)}_{\sim,+[,01]} \cup \{\neg\})$, where $\neg$ — classical negation — is unary.

Here, it is supposed that $\Sigma = \Sigma_{\simeq,+[,01]}$, while $\neg^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i, 1-j\rangle$, for all $i,j \in 2$. Then, one can take $\lambda_{\mathcal{T}}(\{\sim\}\neg) = \{\vdash \{\sim\}x_0\}$ and $\rho_{\mathcal{T}}(\{\sim\}\neg) = \{\{\sim\}x_0 \vdash\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\{\sim\}\neg) = \{\{\{\sim\}x_0, \{\sim\}\neg x_0\} \vdash\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\{\sim\}\neg) = \{\vdash \{\{\sim\}x_0, \{\sim\}\neg x_0\}\}$. Thus, we get:

Corollary 7.8. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CD}_{[01]}$ resulted from $\mathcal{D}_{[01]}$ by adding the following rules:*

$$N_1 \quad \frac{(x_1 \wedge \neg x_1) \vee x_0}{x_0} \qquad N_2 \quad x_0 \vee \neg x_0 \qquad N_3 \quad \frac{(\sim x_1 \wedge \sim\neg x_1) \vee x_0}{x_0} \qquad N_4 \quad \sim x_0 \vee \sim\neg x_0$$

7.2.2. *The bilattice expansions.* Let $\Sigma^{(\supset)}_{\sim/\simeq,2:+[,01]} \triangleq (\Sigma^{(\supset)}_{\sim/\simeq,+[,01]} \cup \{\sqcap, \sqcup\}[\cup\{\mathbf{0},\mathbf{1}\}])$, where $\sqcap$ and $\sqcup$ — *knowledge* conjunction and disjunction, respectively — are binary [while $\mathbf{0}|\mathbf{1}$ — the "under-|over-defined" constant, respectively — are nullary].

Here, it is supposed that $\Sigma = \Sigma_{\sim/\simeq,2:+[,01]}$, while

$$(\langle i,j\rangle(\sqcap|\sqcup)^{\mathfrak{A}}\langle k,l\rangle) \triangleq \langle(\min|\max)(i,k), (\max|\min)(j,l)\rangle,$$

for all $i,j,k,l \in 2$ [whereas $\mathbf{0}^{\mathfrak{A}} \triangleq \mathsf{n}$ and $\mathbf{1}^{\mathfrak{A}} \triangleq \mathsf{b}$].

First, let $\Sigma = \Sigma_{\sim,2:+}$, in which case $\mathcal{A}' = \varnothing$. Then, one can take $\lambda_{\mathcal{T}}(\{\sim\}\sqcap) = \{\{\{\sim\}x_0, \{\sim\}x_1\} \vdash\}$ and $\rho_{\mathcal{T}}(\{\sim\}\sqcap) = \{\vdash \{\sim\}x_0, \vdash \{\sim\}x_1\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\{\sim\}\sqcap) = \{\{\sim\}(x_0 \sqcap x_1) \vdash \{\sim\}x_0, \{\sim\}(x_0 \sqcap x_1) \vdash \{\sim\}x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\{\sim\}\sqcap) = \{\{\{\sim\}x_0, \{\sim\}x_1\} \vdash \{\sim\}(x_0 \sqcap x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\{\sim\}\sqcup) = \{\{\sim\}x_0 \vdash, \{\sim\}x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\{\sim\}\sqcup) = \{\varnothing \vdash \{\{\sim\}x_0, \{\sim\}x_1\}\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\{\sim\}\sqcup) = \{\{\sim\}(x_0 \sqcup x_1) \vdash \{\{\sim\}x_0, \{\sim\}x_1\}\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\{\sim\}\sqcup) = \{\{\sim\}x_0 \vdash \{\sim\}(x_0 \sqcup x_1), \{\sim\}x_1 \vdash \{\sim\}(x_0 \sqcup x_1)\}$. Thus, we get:

Corollary 7.9. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}$ resulted from adding to $\mathcal{D}$ the following rules:*

$$KC \quad \frac{(x_1 \wedge x_2) \vee x_0}{(x_1 \sqcap x_2) \vee x_0}\updownarrow \qquad KD \quad \frac{(x_1 \vee x_2) \vee x_0}{(x_1 \sqcup x_2) \vee x_0}\updownarrow \qquad NKC \quad \frac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \sqcap x_2) \vee x_0}\updownarrow \qquad NKD \quad \frac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \sqcup x_2) \vee x_0}\updownarrow$$

Likewise, let $\Sigma = \Sigma_{\sim,2:+,01}$, in which case $\mathcal{A}'''$ is as above, while $\mathcal{A}' = (\{\bot \vdash, \top\} \cup \{\sim^i\mathbf{0} \vdash, \sim^i\mathbf{1} \mid i \in 2\})$, and so we have:

Corollary 7.10. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}_{01}$ resulted from adding to $\mathcal{BL} \cup \mathcal{D}_{01}$ the axioms $\sim^i\mathbf{1}$ and the rules $\frac{\sim^i\mathbf{0} \vee x_0}{x_0}$, where $i \in 2$.*

Finally, when $\Sigma = \Sigma_{\simeq,2:+[,01]}$, we have:

Corollary 7.11. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CD} \cup \mathcal{BL}_{[01]}$.*

7.2.3. *Implicative expansions.* Here, it is supposed that $\supset \in \Sigma$, while $(\langle i,j\rangle \supset^{\mathfrak{A}} \langle k,l\rangle) \triangleq \langle \max(1-i,k), \max(1-i,l)\rangle$, for all $i,j,k,l \in 2$, in which case $\mathcal{A}$ is $\supset$-implicative, whereas $DB^{\supset}_{4[,01]}$ is defined to be the logic of $\mathcal{DM}^{\supset}_{4[,01]} \triangleq (\mathcal{A}{\restriction}\Sigma^{\supset}_{\sim,+[,01]})$.

First, let $\Sigma = \Sigma^{\supset}_{\sim,+}$. Clearly, one can take $\lambda_{\mathcal{T}}(\sim\supset) = \{\{x_0, \sim x_1\} \vdash\}$ and $\rho_{\mathcal{T}}(\sim\supset) = \{\vdash x_0, \vdash \sim x_1\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim\supset) = \{\sim(x_0 \supset x_1) \vdash x_0, \sim(x_0 \supset x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim\supset) = \{\{x_0, \sim x_1\} \vdash \sim(x_0 \supset x_1)\}$. Therefore, taking Remark 6.1 into account, we get:

Corollary 7.12. *$DB^{\supset}_4$ is axiomatized by the calculus $\mathcal{D}^{\supset}$ resulted from $\mathcal{PC}^{\supset}_{+}$ by adding the following axioms:*

$$\sim\sim x_0 \supset x_0 \qquad x_0 \supset \sim\sim x_0 \tag{7.1}$$

$$\sim(x_0 \vee x_1) \supset \sim x_i \qquad \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \vee x_1)) \tag{7.2}$$

$$\sim x_i \supset \sim(x_0 \wedge x_1) \qquad (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \wedge x_1) \supset x_2)) \tag{7.3}$$

$$\sim(x_0 \supset x_1) \supset \sim^i x_i \qquad x_0 \supset (\sim x_1 \supset \sim(x_0 \supset x_1))$$

where $i \in 2$.

It is remarkable that $\mathcal{D}^{\supset}$ is actually the calculus Par introduced in [14] but regardless to any semantics. In this way, the present study provides a new (and quite immediate) insight into the issue of semantics of Par first being due to [17] but with using the intermediate purely-multi-conclusion sequent calculus $GPar$ actually introduced in [14] regardless to any semantics too and then studied semantically in [17].

Likewise, in case $\Sigma = \Sigma^{\supset}_{\sim,+,01}$, we have:

Corollary 7.13. *$DB^{\supset}_{4,01}$ is axiomatized by the calculus $\mathcal{D}^{\supset}_{01}$ resulted from $\mathcal{D}^{\supset} \cup \mathcal{PC}^{\supset}_{+,01}$ by adding the axioms $\sim\bot$ and $\sim\top \supset x_0$.*

Now, let $\Sigma = \Sigma^{\supset}_{\sim,2:+}$. Then, we have:

Corollary 7.14. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}^{\supset}$ resulted from $\mathcal{D}^{\supset}$ by adding the following axioms:*

$$(x_0 \sqcap x_1) \supset x_i \qquad x_0 \supset (x_1 \supset (x_0 \sqcap x_1))$$

$$x_i \supset (x_0 \sqcup x_1) \qquad (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \sqcup x_1) \supset x_2))$$

$$\sim(x_0 \sqcap x_1) \supset \sim x_i \qquad \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \sqcap x_1))$$

$$\sim x_i \supset \sim(x_0 \sqcup x_1) \qquad (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \sqcup x_1) \supset x_2))$$

where $i \in 2$.

Likewise, when $\Sigma = \Sigma^{\supset}_{\sim,2:+,01}$, we have:

Corollary 7.15. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}^{\supset}_{01}$ resulted from $\mathcal{BL}^{\supset} \cup \mathcal{D}^{\supset}_{01}$ by adding the axioms $\sim^i\mathbf{1}$ and $\sim^i\mathbf{0} \supset x_0$, where $i \in 2$.*

Further, let $\Sigma = \Sigma^{\supset}_{\simeq,+[,01]}$. Then, taking (4.12) and Corollary (4.16)(i) into account, we have:

Corollary 7.16. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}^{\supset}_{[01]}$ resulted from $\mathcal{D}^{\supset}_{[01]}$ by adding the axioms N_2, N_4 and $\sim^i x_1 \supset (\sim^i\neg x_i \supset x_0)$, where $i \in 2$.*

Finally, when $\Sigma = \Sigma^{\supset}_{\simeq,2:+[,01]}$, we have:

Corollary 7.17. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}^{\supset} \cup \mathcal{BL}^{\supset}_{[01]}$.*

7.2.4. *Disjunctive extensions.* Let $\mathcal{L} = \mathcal{L}_{\mathcal{A}}$, $\mathcal{L}^{\mathcal{C}} \triangleq \mathcal{L}_{\mathcal{L}\cup\mathcal{C}}$, where $\mathcal{C}$ is a Σ-calculus, and $A_{[\not{\mathsf{b}}]([,]\not{\mathsf{n}})} \triangleq (A[\setminus\{\mathsf{b}\}](\setminus\{\mathsf{n}\}))$. Though $\mathcal{A}$ has two distinct non-distinguished values f and n, we have the following partial analogue of Lemma 2.2 of :

Lemma 7.18. *$\mathcal{L}$ is $\sqsupset$-implicative iff $\mathcal{A}$ is so.*

Proof. The "if" part is immediate. Conversely, assume $\mathcal{L}$ is $\sqsupset$-implicative, in which case, by Lemma 4.15, it is $\veebar_{\sqsupset}$-disjunctive, and so, being $\vee$-disjunctive, for $\mathcal{A}$ is so, contains $(x_0 \veebar_{\sqsupset} x_1) \vdash (x_0 \vee x_1)$ (in particular, by (4.13), it contains $(x_0 \sqsupset x_1) \vee x_0$). In this way, (4.12), (4.14) and the $\vee$-disjunctivity of $\mathcal{A}$ complete the argument. $\square$

Next, under the identification of submatrices of $\mathcal{A}$ with the carriers of their underlying algebras we follow below tacitly (in which case relatively hereditary subclasses of $\mathbf{S}_*(\mathcal{A})$ become actually lower cones of it, and so for finding all former ones it suffices to find all anti-chains of it), $\mathbf{S}_*(\mathcal{A}) \subseteq \mathbf{S}_*(\mathcal{DM}_4) = \{A, A_{\not{\mathsf{b}}}, A_{\not{\mathsf{n}}}, A_{\not{\mathsf{b}},\not{\mathsf{n}}}, \{\mathsf{n}\}\} = (\mathbf{S}_{[*]}(\mathcal{DM}_{4,01}) \cup \{\mathsf{n}\})$, in which case $\{A_{\not{\mathsf{b}}}, A_{\not{\mathsf{n}}}\}$ and $\{A_{\not{\mathsf{n}}}, \{\mathsf{n}\}\}$ are the only non-one-element anti-chains of $\mathbf{S}_*(\mathcal{A}) \setminus \{\mathcal{A}_{\not{\mathsf{b}},\not{\mathsf{n}}}\})$, while $\mathbf{S}_*(\mathcal{A}_{\not{\mathsf{b}},\not{\mathsf{n}}}) = (\mathbf{S}_*(\mathcal{A}_{\not{\mathsf{b}}}) \cap \mathbf{S}_*(\mathcal{A}_{\not{\mathsf{n}}}))$, whereas $\mathbf{S}_*(\mathcal{B})$, where $\mathcal{B} \in (\mathbf{S}_*(\mathcal{A}) \setminus \{\mathcal{A}_{\not{\mathsf{b}},\not{\mathsf{n}}}\})$, is relatively axiomatized, according to the constructive proof of Lemma 5.6, as follows. First, if $B = \{\mathsf{n}\}$, then, for each $\mathcal{C} \in (\mathbf{S}_*(\mathcal{A}) \setminus \mathbf{S}(\mathcal{B}))$, $c \triangleq \mathsf{t} \in (C \setminus B)$, in which case $\Phi_{\mathcal{C},c} = (x_0 \vdash)$, and so this is a relative axiomatization of $\mathbf{S}_*(\mathcal{B})$. Likewise, if $B = A_{\not{\mathsf{b}}|\not{\mathsf{n}}}$, then, for each $\mathcal{C} \in (\mathbf{S}_*(\mathcal{A}) \setminus \mathbf{S}(\mathcal{B}))$, $c \triangleq (\mathsf{b}|\mathsf{n}) \in (C \setminus B)$, in which case $\Phi_{\mathcal{C},c} = ((0|1) : \{x_0, \sim x_0\})$, and so this is a relative axiomatization of $\mathbf{S}_*(\mathcal{B})$. In this way, taking (4.14) and Lemma 7.18 into account, we eventually get:

Theorem 7.19. *$\vee$-disjunctive[(/$\sqsupset$-implicative/axiomatic)] extensions of $\mathcal{L}$ [having axioms (more specifically, being $\sqsupset$-implicative)] form an image of the* $(9[-3])$*-element poset of all $\vee$-disjunctive extensions of $DB_{4[,01]}$ depicted at Figure 1 [with merely solid circles], where:*

$$\vdash x_0 \vee \sim x_0, \tag{7.4}$$

$$x_1 \vdash x_0 \vee \sim x_0, \tag{7.5}$$

$$\vdash x_0, \tag{7.6}$$

$$x_1 \vdash x_0, \tag{7.7}$$

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash x_0, \tag{7.8}$$

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash (x_2 \vee \sim x_2) \vee x_0, \tag{7.9}$$

$$\vdash (\sim x_1 \sqsupset (x_1 \sqsupset x_0), \tag{7.10}$$

$$\vdash (\sim x_1 \sqsupset (x_1 \sqsupset (x_0 \vee \sim x_0)), \tag{7.11}$$

and are "relatively axiomatized"|"defined" by the "(axiomatic) Σ-calculi"|"anti-chains of $\mathbf{S}_*(\mathcal{A})$ being the intersections of this and the anti-chains of $\mathbf{S}_*(\mathcal{DM}_{4[,01]})$" marking corresponding nodes, in which case different nodes may correspond to just different relative axiomatizations of same $\vee$-disjunctive[(/$\sqsupset$-implicative/axiomatic)] extensions of $\mathcal{L}$.

In case $\Sigma = \Sigma_{\sim,+}$, Theorem 7.19 subsumes both Corollary 5.3 of [15] and, in view of Theorem 4.1 therein, the reference [Pyn 95 a] of [16] as well as shows both that Kleene's three-valued logic [7] is the extension of DB_4 relatively axiomatized by the *Resolution* (cf. [25] for roots of this terminology) rule (7.8), and, collectively with Theorem 4.13 of [18], that $\Re_{\veebar}((\tau_{\veebar}[)\mathcal{S}([]))$ cannot be replaced by $(\tau_{\veebar}[)\mathcal{S}([])^{\setminus 1}$ in the formulation(s) of Lemma 5.1 (resp., Corollaries 4.3, 5.7 and Theorem 5.4), when taking $\mathcal{S} = \{i : \{x_0, \sim x_0\} \mid i \in 2\}$. Likewise, in case $\Sigma = \Sigma^{\supset}_{\sim,+[,01]}$ (cf. Subsubsection 7.2.3), Theorem 7.19 with $\sqsupset = \supset$ subsumes Corollary 5.4 of [24]. And what is more, in case $\Sigma = \Sigma^{\supset}_{\sim,+}$, Theorem 7.19 shows that the calculus *PCont*,

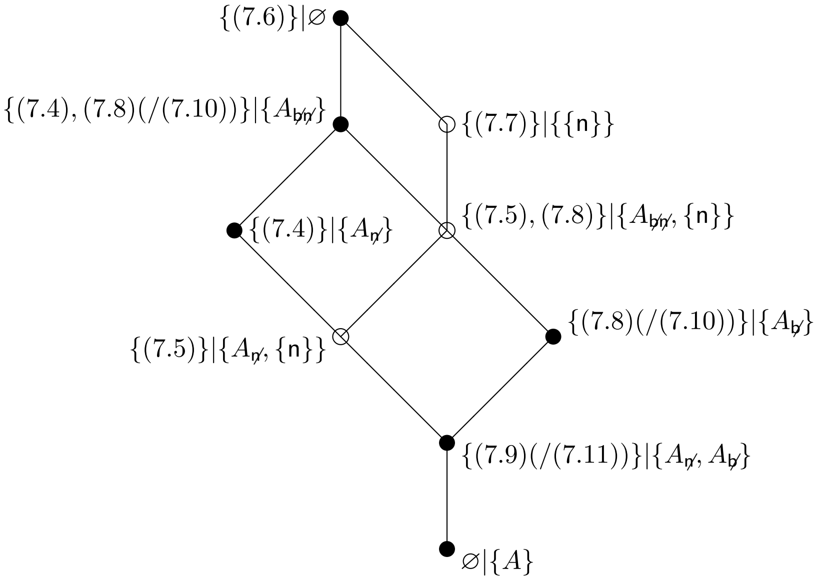

FIGURE 1. The poset of $\vee$-disjunctive(/$\supset$-implicative/axiomatic) extensions of $DB^{(\supset)}_{4[,01]}$ [with merely solid circles] (with merely solid circles and $\sqsupset{=}\supset$) and their "relative axiomatizations"|"defining anti-chains of $\mathbf{S}_*(\mathcal{DM}^{(\supset)}_{4[,01]})$".

resulted from $GPar = \mathfrak{D}^{\supset}$ by adding (7.4) and introduced in [14] regardless to any semantics as well as, axiomatizes the *logic of antinomies LA* [1] being defined by $A_{\not\mathsf{n}}$. Concluding this Subsubsection, we discuss other two representative classes of expansions of DB_4 involved above as well as in [17, 24] and being rectangular to one another in a sense.

7.2.4.1. Classically-negative expansions. Here, it is supposed that $\Sigma \supseteq \Sigma_{\simeq,+}$ (cf. Subsubsection 7.2.1), in which case $\mathcal{A}$ is $\sqsupset$-implicative, where $(x_0 \sqsupset x_1) \triangleq (\neg x_0 \vee x_1)$, while $\mathbf{S}_{\{*\}}(\mathcal{A}) = \{A[, A_{\not\mathsf{b}\not\mathsf{n}}]\}$, and so we get:

Corollary 7.20 (cf. Corollary 5.1(i) of [24])**.** *$\mathcal{L}$ has no proper consistent $\vee$-disjunctive/$\sqsupset$-implicative/axiomatic extension, if $A_{\not\mathsf{b}\not\mathsf{n}}$ does not form a subalgebra of $\mathfrak{A}$, and has a unique one, otherwise, in which case this is equal to $\mathcal{L}^{(7.4)} = \mathcal{L}^{(7.9)} = \mathcal{L}_{A_{\not\mathsf{b}\not\mathsf{n}}}$, while $\mathcal{L}^{(7.8)|(7.10)}$ is inconsistent.*

7.2.4.2. Bilattice expansions. Here, it is supposed that $\Sigma \supseteq \Sigma_{\sim,2:+}$ (cf. Subsubsection 7.2.2), in which case $\mathbf{S}_*(\mathcal{A}) = \{A[, \{\mathsf{n}\}]\}$, and so we get:

Corollary 7.21 (cf. Corollary 5.2 of [24])**.** *$\mathcal{L}$ has no proper consistent $\vee$-disjunctive extension, if $\{\mathsf{n}\}$ does not form a subalgebra of $\mathfrak{A}$, and has a unique one, otherwise, in which case this is equal to $\mathcal{L}^{(7.8)} = \mathcal{L}^{(7.9)} = \mathcal{L}^{(7.7)} = \mathcal{L}_{\{\mathsf{n}\}}$, and so has no axiom, while $\mathcal{L}^{(7.4)}$ is inconsistent. In particular, $\mathcal{L}$ has no proper consistent axiomatic extension.*

7.3. **Łukasiewicz' finitely-valued logics.** Let $\Sigma \triangleq \{\supset, \neg\}$, $n \in (\omega \setminus 2)$ and $\mathcal{L}_n$ the Σ-matrix with $L_n \triangleq (n \div (n-1))$, $D^{\mathcal{L}_n} \triangleq \{1\}$, $\neg^{\mathfrak{L}_n} a \triangleq (1-a)$ and $(a \supset^{\mathfrak{L}_n} b) \triangleq \min(1, 1-a+b)$, for all $a, b \in L_n$. The logic Ł$_n$ of $\mathcal{L}_n$ is known as *Łukasiewicz' n-valued logic* [11] (cf. [9] for the three-valued case alone though). By induction on any $m \in (\omega \setminus 1)$, define the secondary unary connective $m\otimes$ of Σ as follows:

$$(m \otimes x_0) \triangleq \begin{cases} x_0 & \text{if } m = 1, \\ \neg x_0 \supset ((m-1) \otimes x_0) & \text{otherwise,} \end{cases}$$

in which case $(m \otimes^{\mathfrak{L}_n} a) = \min(1, m \cdot a)$, for all $a \in L_n$, and so, in particular, $(m\otimes)^{\mathfrak{L}_n}$ is $\leqslant$-monotonic. Then, set $(\Box x_0) \triangleq (\neg^{\min(1,n-2)}(n-1) \otimes \neg^{\min(1,n-2)} x_0)$ and $(x_0 \sqsupset x_1) \triangleq (\Box x_0 \supset \Box x_1)$, being secondary, unless $n = 2$, when $(\Box x_0) = x_0$, and so $\sqsupset = \supset$ is primary. In that case, $\Box^{\mathfrak{L}_n} = ((((n-1) \div (n-1)) \times \{0\}) \cup \{\langle 1, 1\rangle\})$, and so $\mathcal{L}_n$ is $\sqsupset$-implicative, for $(\mathcal{L}_n \restriction 2) = \mathcal{L}_2$ is $\supset$-implicative.

According to the constructive proof of Proposition 6.10 of [20], for each $i \in ((n-1) \setminus 2)$, there is some $\iota_i \in \mathrm{Tm}^1_{\{\neg, 2\otimes\}}$ such that $(\iota_i^{\mathfrak{L}_n}(\frac{i}{n-1}) = 1) \Leftrightarrow (\iota_i^{\mathfrak{L}_n}(\frac{i-1}{n-1}) \neq 1)$. In addition, put $\iota_{n-1} \triangleq x_0 \in \mathrm{Tm}^1_{\{\neg, 2\otimes\}}$ and, in case $n \neq 2$, $\iota_1 \triangleq \neg x_0 \in \mathrm{Tm}^1_{\{\neg, 2\otimes\}}$. In this way, for each $i \in (n \setminus 1)$, it holds that $(\iota_i^{\mathfrak{L}_n}(\frac{i}{n-1}) = 1) \Leftrightarrow (\iota_i^{\mathfrak{L}_n}(\frac{i-1}{n-1}) \neq 1)$. On the other hand, for every $\iota \in \mathrm{Tm}^1_{\{\neg, 2\otimes\}}$, $\iota^{\mathfrak{L}_n}$ is either $\leqslant$-monotonic or $\leqslant$-anti-monotonic, for both $x_0^{\mathfrak{L}_n} = \eth_n$ and $(2\otimes)^{\mathfrak{L}_n}$ are $\leqslant$-monotonic, while $\neg^{\mathfrak{L}_n}$ is $\leqslant$-anti-monotonic. Therefore, for each $i \in N_{0/1} \triangleq \{j \in (n \setminus 1) \mid \iota_j^{\mathfrak{L}_n}(\frac{j}{n-1}) = / \neq 1\}$, $\iota_i^{\mathfrak{L}_n}$ is $\leqslant$-monotonic/-anti-monotonic, in which case $(\iota_i^{\mathfrak{L}_n})^{-1}[\{1\}] = (((n \setminus i) \div (n-1))/(i \div (n-1)))$, respectively, and so $\Im \triangleq (\mathrm{img}\, \bar{\iota}) \supseteq (\{x_0\} \cup \{\neg x_0 \mid n \neq 2\})$ is a finite equality determinant for $\mathcal{L}_n$, $\bar{\iota}$ being injective, in which case $\neg \in \Im$, unless $n = 2$, when all $\Im$-compound connectives are not in $\Im = \mathrm{Var}_1$. And what is more, as it follows from the constructive proof of Proposition 6.10 of [20], $\Im$-compound connectives of Σ belonging to $\Im$ other than $\neg$ are exactly those of the form $\iota_i(\neg)$, where $\frac{n-1}{2} \geqslant i \in (n \setminus 2)$, and so an $\Im$-compound connective of Σ of the form $(\iota_i(\neg)$, where $i \in (n \setminus 1)$, is not in $\Im$ iff $i \in N_c \triangleq \{j \in ((n - \min(1, n-2)) \setminus 1) \mid (j \neq 1) \Rightarrow ((n-1) \in (2 \cdot j))\}$. In particular, in case $n \in (5 \setminus 3)$, $\neg$ is the only $\Im$-compound connective of Σ belonging to $\Im$. As $(N_0 \cap N_1) = \varnothing$ and $(N_0 \cup N_1) = (n \setminus 1)$, we have the mapping $\mu \triangleq \{\langle i, k\rangle \in ((n \setminus 1) \times 2) \mid i \in N_k\} : (n \setminus 1) \to 2$.

Let $\mathcal{A} \triangleq \mathcal{L}_n$. Then, $\mathcal{A}' = \varnothing$. Moreover, under the conventions adopted in both [22] and [23], we see that both

$$\begin{aligned} \{I_{i-1} : \varphi\} &\leftrightarrow (\mu(i) : \iota_i(\varphi)), \\ \{F_i : \varphi\} &\leftrightarrow ((1-\mu(i)) : \iota_i(\varphi)), \end{aligned}$$

where $i \in (n \setminus 1)$ and $\varphi \in \mathrm{Fm}_\Sigma$, are true in $\mathcal{A}$. Hence, in view of Corollary 2.4 of [22], $\mathcal{A}''_{\eth} = \{((1-\mu(i)) : \iota_i) \uplus (\mu(j) : \iota_j) \mid i, j \in (n \setminus 1), i \in j\}$. And what is more, in view of Lemma 2.1 of [23], we have the Σ-sequent $\Im$-table $\mathcal{T}$ for $\mathcal{A}$ given as follows. First, for all $i \in N_c$ and all $m \in 2$, let $\pi_m(\mathcal{T})(\iota_i(\neg)) \triangleq \{(1-)^{\mu(i)}(1-)^m(1-\mu(n-i)) : \iota_{n-i}\}$. Next, for all $i \in (n \setminus 1)$, let $\pi_{1-\mu(i)}(\mathcal{T})(\iota_i(\supset)) \triangleq \{(\mu(n-1-k) : \nu_{n-1-k}) \uplus ((1-\mu(i-k)) : \nu_{i-k}(x_1)) \mid k \in i\}$ and $\pi_{\mu(i)}(\mathcal{T})(\iota_i, \supset) \triangleq (\{\{((1-\mu(n-k)) : \iota_{n-k}) \uplus (\mu(i-k) : \iota_{i-k}(x_1)) \mid k \in (i \setminus 1)\} \cup \{1-\mu(n-i)) : \iota_{n-i}, \mu(i) : \iota_i(x_1)\}\})$. In this way, we eventually get:

Corollary 7.22. Ł_n *is axiomatized by the finite calculus* $\mathcal{L}_n$ *resulted from* $\mathfrak{I}^{\mathrm{PL}}_{\sqsupset}$ *by adding the following axioms:*

$$\iota_i \sqsupset \iota_j \qquad (\langle i, j\rangle \in ((\ker \mu) \cap (\in \cap\, n^2)^{(2\cdot\mu(i))-1})$$

$$\iota_i \veebar_{\sqsupset} \iota_j \qquad (\langle i, j\rangle \in (\mu^{-1}[\in \cap\, 2^2] \cap (\in \cap\, n^2))$$

$$\iota_i \sqsupset (\iota_j \sqsupset x_1) \qquad (\langle i, j\rangle \in (\mu^{-1}[\ni \cap\, 2^2] \cap (\in \cap\, n^2))$$

$$\iota_{n-i} \veebar_{\sqsupset} \iota_i(\neg x_0) \qquad (i \in N_c, \mu(i) = \mu(n-i))$$

$$\iota_{n-i} \sqsupset (\iota_i(\neg x_0) \sqsupset x_1) \qquad (i \in N_c, \mu(i) = \mu(n-i))$$

$$\iota_{n-i} \sqsupset \iota_i(\neg x_0) \qquad (i \in N_c, \mu(i) \neq \mu(n-i))$$

$$\iota_i(\neg x_0) \sqsupset \iota_{n-i} \qquad (i \in N_c, \mu(i) \neq \mu(n-i))$$

$$\iota_{n-1-k} \sqsupset (\iota_{i-k}(x_1) \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 0 \neq \mu(i-k))$$

$$\iota_{n-1-k} \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset \iota_{i-k}(x_1)) \qquad (n \neq 2, k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 0 = \mu(i-k))$$

$$\iota_{n-1-k} \sqsupset (\iota_{i-k}(x_1) \sqsupset \iota_i(x_0 \supset x_1)) \qquad (k \in i \in (n \setminus 1), \mu(i) \neq \mu(n-1-k) = 0 \neq \mu(i-k))$$

$$\iota_{i-k}(x_1) \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset \iota_{n-1-k}) \qquad (k \in i \in (n \setminus 1), \mu(i) = 0 \neq \mu(n-1-k) = \mu(i-k))$$

$$(\iota_{n-1-k} \veebar_{\sqsupset} \iota_{i-k}(x_1)) \veebar_{\sqsupset} \iota_i(x_0 \supset x_1) \qquad (k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 1 \neq \mu(i-k))$$

$$(\iota_{n-1-k} \sqsupset x_2) \sqsupset ((\iota_{i-k}(x_1) \sqsupset x_2) \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) = 0 = \mu(i-k) \neq \mu(n-1-k))$$

$$(\iota_{n-1-k} \sqsupset x_2) \sqsupset ((\iota_i(x_0 \supset x_1) \sqsupset x_2) \sqsupset (\iota_{i-k}(x_1) \sqsupset x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) = 1 = \mu(n-1-k) = \mu(i-k))$$

$$(\iota_{i-k}(x_1) \sqsupset x_2) \sqsupset ((\iota_i(x_0 \supset x_1) \sqsupset x_2) \sqsupset (\iota_{n-1-k} \sqsupset x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) \neq 0 = \mu(n-1-k) = \mu(i-k))$$

$$\iota_{n-k} \sqsupset (\iota_{i-k}(x_1) \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 1 \neq \mu(i-k))$$

$$\iota_{n-k} \sqsupset (\iota_{i-k}(x_1) \sqsupset \iota_i(x_0 \supset x_1)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 1 \neq \mu(i-k))$$

$$\iota_{n-k} \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset \iota_{i-k}(x_1)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 1 = \mu(i-k))$$

$$\iota_{i-k}(x_1) \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset \iota_{n-k}) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 0 = \mu(i-k))$$

$$(\iota_{n-k} \veebar_{\sqsupset} \iota_{i-k}(x_1)) \veebar_{\sqsupset} \iota_i(x_0 \supset x_1) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 0 \neq \mu(i-k))$$

$$(\iota_{n-k} \sqsupset x_2) \sqsupset ((\iota_{i-k}(x_1) \sqsupset x_2) \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 0 \neq \mu(i-k))$$

$$(\iota_{n-k} \sqsupset x_2) \sqsupset ((\iota_i(x_0 \supset x_1) \sqsupset x_2) \sqsupset (\iota_{i-k}(x_1) \sqsupset x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 0 = \mu(i-k))$$

$$(\iota_{i-k}(x_1) \sqsupset x_2) \sqsupset ((\iota_i(x_0 \supset x_1) \sqsupset x_2) \sqsupset (\iota_{n-k} \sqsupset x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 1 = \mu(i-k))$$

$$\iota_{n-i} \sqsupset \iota_i(x_0 \supset x_1) \qquad (i \in N_0 \not\ni (n-i))$$

$\iota_i(x_0 \supset x_1) \sqsupset \iota_{n-i}$ $\quad (i \in N_1 \not\ni (n-i))$

$\iota_{n-i} \sqsupset (\iota_i(x_0 \supset x_1) \sqsupset x_2)$ $\quad (i \in N_1 \ni (n-i))$

$\iota_{n-i} \veebar_{\sqsupset} \iota_i(x_0 \supset x_1)$ $\quad (n \neq 2, i \in N_0 \ni (n-i))$

$\iota_i(x_1) \sqsupset \iota_i(x_0 \supset x_1)$ $\quad (n \neq 2, i \in N_0)$

$\iota_i(x_0 \supset x_1) \sqsupset \iota_i(x_1)$ $\quad (i \in N_1)$

It is remarkable that, in the classical case, when $n = 2$, the additional axioms of $\mathcal{L}_n$ are exactly the Excluded Middle Law axiom $(x_0 \veebar_{\sqsupset} \neg x_0) = ((x_0 \supset \neg x_0) \supset \neg x_0)$ and the Ex Contradictione Quodlibet axiom $x_0 \supset (\neg x_0 \supset x_1)$, $\mathcal{L}_2$ being a well-known natural Hilbert-style axiomatization of the classical logic. And what is more, $\mathcal{L}_n$ grows just polynomially (more precisely, quadratically) on n, so it eventually looks relatively good, the additional axioms of $\mathcal{L}_3$ being as follows, where $i \in 2$:

$$\begin{array}{lll} \neg x_1 \sqsupset (x_1 \sqsupset x_0) & \neg^i x_i \sqsupset ((x_0 \supset x_1) \sqsupset \neg^i x_{1-i}) & \neg x_0 \sqsupset (x_0 \supset x_1) \\ x_0 \sqsupset \neg\neg x_0 & x_0 \sqsupset (\neg x_1 \sqsupset \neg(x_0 \supset x_1)) & x_1 \sqsupset (x_0 \supset x_1) \\ \neg\neg x_0 \sqsupset x_0 & (x_0 \veebar_{\sqsupset} \neg x_1) \veebar_{\sqsupset} (x_0 \supset x_1) & \neg(\neg x_0 \supset x_1) \sqsupset \neg x_1 \end{array}$$

Concluding this discussion, we should like to highlight that, though, in general, an analytical expression (if any, at all) for $\bar{\iota}$ has not been known yet, the constructive proof of Proposition 6.10 of [20] has been implemented upon the basis of SCWI-Prolog resulting in a quite effective logical program (taking less than second up to $n = 1000$) calculating $\bar{\iota}$, and so immediately yielding definitive explicit formulations of both $\mathcal{T}$ (in particular, of the Gentzen-style axiomatization $\mathcal{S}^{(0,0)}_{\mathcal{A},\mathcal{T}}$ of Ł$_n$; cf. [19]) and the Hilbert-style axiomatization $\mathcal{L}_n$ of Ł$_n$ found above. It is also remarkable that our deductive approach seems to be convergent with (though not *absolutely* identical to) the well-known one [28].

7.4. **Hałkowska-Zajac logic.** Here, it is supposed that $\Sigma \triangleq \Sigma_{\sim,+}$, $(\mathfrak{A}{\restriction}\Sigma_+) \triangleq \mathfrak{D}_3$, $\sim^{\mathfrak{A}} i \triangleq (\min(1,i)\cdot(3-i))$, for all $i \in 3$, and $D^{\mathcal{A}} \triangleq \{0,2\}$, in which case $\mathcal{A}$, defining the logic HZ [6], is $\supset$-implicative, where $(x_0 \supset x_1) \triangleq ((\sim x_0 \wedge \sim x_1) \vee x_1)$ is secondary, while $\{x_0, \sim x_0\}$ is an equality determinant for $\mathcal{A}$ (cf. Example 2 of [19]), and so $\mathcal{A}' = \varnothing$ and $\mathcal{A}'_{\{\eth\}} = \{\vdash \{\sim x_0, x_0\}\}$. First, we have $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = a$, for all $a \in A$. Therefore, one can take $\lambda_{\mathcal{T}}(\sim\sim) = \{x_0 \vdash\}$ and $\rho_{\mathcal{T}}(\sim\sim) = \{\vdash x_0\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim\sim) = \{\sim\sim x_0 \vdash x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim\sim) = \{x_0 \vdash \sim\sim x_0\}$. Next, consider any $a, b \in A$. Then, $\sim^{\mathfrak{A}}(a(\wedge/\vee)^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff either/both $\sim^{\mathfrak{A}} a \in D^{\mathcal{A}}$ or/and $\sim^{\mathfrak{A}} b \in D^{\mathcal{A}}$. Therefore, one can take $\lambda_{\mathcal{T}}(\sim\vee) = \{\{\sim x_0, \sim x_1\} \vdash\}$ and $\rho_{\mathcal{T}}(\sim\vee) = \{\vdash \sim x_0, \vdash \sim x_1\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim\vee) = \{\sim(x_0 \vee x_1) \vdash \sim x_0, \sim(x_0 \vee x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim\vee) = \{\{\sim x_0, \sim x_1\} \vdash \sim(x_0 \vee x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim\wedge) = \{\sim x_0 \vdash, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim\wedge) = \{\vdash \{\sim x_0, \sim x_1\}\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim\wedge) = \{\sim(x_0 \wedge x_1) \vdash \{\sim x_0, \sim x_1\}\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim\wedge) = \{\sim x_0 \vdash \sim(x_0 \wedge x_1), \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Moreover, $(a(\wedge/\vee)^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff both $(a = 1) \Rightarrow (b = (0/2))$ and $(b = 1) \Rightarrow (a = (0/2))$. Therefore, one can take $\rho_{\mathcal{T}}(\wedge) = \{\vdash \{x_0, x_1\}, \vdash \{\sim x_0, x_1\}, \vdash \{\sim x_1, x_0\}\}$ and $\lambda_{\mathcal{T}}(\wedge) = \{\{x_0, x_1\} \vdash, \{x_0, \sim x_0\} \vdash \{x_1, \sim x_1\} \vdash\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\wedge) = \{(x_0 \wedge x_1) \vdash \{x_0, x_1\}, (x_0 \wedge x_1) \vdash \{\sim x_0, x_1\}, (x_0 \wedge x_1) \vdash \{\sim x_1, x_0\}\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\wedge) = \{\{x_0, x_1\} \vdash (x_0 \wedge x_1), \{x_0, \sim x_0\} \vdash (x_0 \wedge x_1), \{x_1, \sim x_1\} \vdash (x_0 \wedge x_1)\}$. Likewise, one can take $\rho_{\mathcal{T}}(\vee) = \{\vdash \{x_0, x_1\}, \sim x_1 \vdash x_0, \sim x_0 \vdash x_1\}$ and $\lambda_{\mathcal{T}}(\vee) = \{\{x_0, x_1\} \vdash, \vdash \sim x_0, \vdash \sim x_1\}$ to satisfy (6.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\vee) = \{(x_0 \vee x_1) \vdash \{x_0, x_1\}, \{\sim x_1, (x_0 \vee x_1)\} \vdash x_0, \{\sim x_0, (x_0 \vee x_1)\} \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\vee) = \{\{x_0, x_1\} \vdash (x_0 \vee x_1), \vdash \{\sim x_0, (x_0 \vee x_1)\}, \vdash \{\sim x_1, (x_0 \vee x_1)\}\}$. In this way, we eventually get:

Corollary 7.23. *HZ is axiomatized by the calculus $\mathcal{HZ}$ resulted from $\mathcal{I}^{\mathrm{PL}}_{\supset}$ by adding the axioms* (7.1)*,* (7.2)*,* (7.3) *and the following ones, where $i \in 2$:*

$$(x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \wedge x_1) \supset x_2)) \qquad x_0 \supset (x_1 \supset (x_0 \wedge x_1))$$

$$(\sim x_i \supset x_2) \supset ((x_{1-i} \supset x_2) \supset ((x_0 \wedge x_1) \supset x_2)) \qquad x_i \supset (\sim x_i \supset (x_0 \wedge x_1))$$
$$(x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \vee x_1) \supset x_2)) \qquad x_0 \supset (x_1 \supset (x_0 \vee x_1))$$
$$(\sim x_i \supset (x_0 \vee x_1)) \supset (x_0 \vee x_1) \qquad \sim x_{1-i} \supset ((x_0 \vee x_1) \supset x_i)$$
$$(\sim x_0 \supset x_0) \supset x_0$$

In this connection, recall that an *infinite* Hilbert-style axiomatization of HZ has been due to [29].

8. Conclusions

As a matter of fact, Subsection 7.2 has provided finite Hilbert-style axiomatizations of *all* miscellaneous expansions of DB_4 studied in [17] as well as their disjunctive extensions (in this connection, it is remarkable that we have avoided any guessing their relative axiomatizations right — though such would not be difficult, as it has originally been done in the reference [Pyn 95 a] of [16] — but rather have just followed the constructive proof of Lemma 5.6 to demonstrate its practical applicability to effective/computational finding "good" relative axiomatizations in other more complicated cases like Łukasiewicz' logics). Even though Section 7 does not exhaust *all* interesting applications of Sections 5 and 6, it has definitely incorporated *most acute* ones.

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