



The Reimann Hypothesis

Frank Vega

EasyChair preprints are intended for rapid dissemination of research results and are integrated with the rest of EasyChair.

November 17, 2020

The Riemann Hypothesis

Frank Vega

Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all sufficiently large n , where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. In 2002, Lagarias proved that if the inequality $\sigma(n) \leq H_n + \exp(H_n) \times \log H_n$ holds for all $n \geq 1$, then the Riemann Hypothesis is true, where H_n is the n^{th} harmonic number. In this work, we show certain properties of these both inequalities that leave us to a proof of the Riemann Hypothesis which could be checked by computer.

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n [1]:

$$\sum_{d|n} d.$$

Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and \log is the natural logarithm. Let H_n be $\sum_{j=1}^n \frac{1}{j}$. Say Lagarias(n) holds provided

$$\sigma(n) \leq H_n + \exp(H_n) \times \log H_n.$$

The importance of these properties is:

Theorem 1.1 *If Robins(n) holds for all $n > 5040$, then the Riemann Hypothesis is true [4]. If Lagarias(n) holds for all $n \geq 1$, then the Riemann Hypothesis is true [4].*

It is known that Robins(n) and Lagarias(n) hold for many classes of numbers n . We know this:

Lemma 1.2 *If Robins(n) holds for some $n > 5040$, then Lagarias(n) holds [4].*

2010 Mathematics Subject Classification: Primary 11M26; Secondary 11A41, 26D15.

Keywords: number theory, inequality, sum-of-divisors function, harmonic number, prime.

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [1]. $\text{Robins}(n)$ holds for all $n > 5040$ that are square free [1]. Let $\text{core}(n)$ denotes the square free kernel of a natural number n [1]. We can show this:

Theorem 1.3 *Let $\frac{\pi^2}{6} \times \log \log \text{core}(n) \leq \log \log n$ for some $n > 5040$. Then $\text{Robins}(n)$ holds.*

Moreover, we prove our main theorems:

Theorem 1.4 *$\text{Robins}(n)$ holds for all $n > 5040$ when a prime number $q_m \nmid n$ for $q_m \leq 113$.*

Theorem 1.5 *Let $n > 5040$ and $n = r \times q_m$, where $q_m \geq 113$ denotes the largest prime factor of n . We have checked by computer, if $\text{Lagarias}(r)$ holds, then $\text{Lagarias}(n)$ holds.*

In this way, we finally conclude that

Theorem 1.6 *$\text{Lagarias}(n)$ holds for all $n \geq 1$ and thus, the Riemann Hypothesis is true.*

Proof Every possible counterexample in $\text{Lagarias}(n)$ for $n > 5040$ must have that its greatest prime factor q_m complies with $q_m \geq 113$ because of lemma 1.2 and theorem 1.4. In addition, $\text{Lagarias}(n)$ has been checked for all $n \leq 5040$ by computer. Moreover, for all $n > 5040$ we have that $\text{Lagarias}(n)$ has been recursively verified when its greatest prime factor q_m complies with $q_m \geq 113$ due to theorems 1.4 and 1.5. In conclusion, we show that $\text{Lagarias}(n)$ holds for all $n \geq 1$ and therefore, the Riemann Hypothesis is true. ■

2 Known Results

We use that the following are known:

Lemma 2.1 *From the reference [1]:*

$$f(n) < \prod_{p|n} \frac{p}{p-1}.$$

Lemma 2.2 *From the reference [2]:*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}.$$

Lemma 2.3 *From the reference [4]:*

$$\log(e^\gamma \times (n+1)) \geq H_n \geq \log(e^\gamma \times n).$$

3 A Central Lemma

Lemma 3.1 Given a natural number

$$n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m}$$

such that q_1, q_2, \dots, q_m are prime numbers and a_1, a_2, \dots, a_m are natural numbers, then we obtain the following inequality

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Proof From the lemma 2.1, we know

$$f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

We can easily prove

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} = \prod_{i=1}^m \frac{1}{1 - q_i^{-2}} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

However, we know

$$\prod_{i=1}^m \frac{1}{1 - q_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}}$$

where q_j is the j^{th} prime number and

$$\prod_{j=1}^{\infty} \frac{1}{1 - q_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of lemma 2.2. Consequently, we obtain

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}$$

and thus,

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

■

4 A Particular Case

We prove the Robin's inequality for this specific case:

Lemma 4.1 Given a natural number

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$$

such that $a_1, a_2, a_3, a_4 \geq 0$ are integers, then $\text{Robins}(n)$ holds for $n > 5040$.

Proof Given a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that q_1, q_2, \dots, q_m are prime numbers and a_1, a_2, \dots, a_m are natural numbers, we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1. Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^\gamma \times \log \log(5040) < e^\gamma \times \log \log n$$

and therefore, the proof is completed for that case. Hence, we only need to prove the Robin's inequality is true for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^\gamma \times \log \log(7^7) \approx 4.65.$$

However, we know for $n > 5040$ and $7^7 \mid n$ such that

$$e^\gamma \times \log \log(7^7) \leq e^\gamma \times \log \log n$$

and as a consequence, the proof is completed. ■

5 A Condition on $\text{core}(n)$

Theorem 5.1 Let $\frac{\pi^2}{6} \times \log \log \text{core}(n) \leq \log \log n$ for some $n > 5040$. Then $\text{Robins}(n)$ holds.

Proof We will check the Robin's inequality for a natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that q_1, q_2, \dots, q_m are prime numbers and a_1, a_2, \dots, a_m are natural numbers. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n$$

according to the lemma 3.1. We obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^\gamma \times \log \log n$$

where $n' = q_1 \times \cdots \times q_m$ is the $\text{core}(n)$ [1]. However, $\text{Robins}(n')$ has been proved for all the square free integers $n' \notin \{2, 3, 5, 6, 10, 30\}$ [1]. In addition, due to the lemma 4.1, we know $\text{Robins}(n)$ holds for all $n > 5040$ when $\text{core}(n) \in \{2, 3, 5, 6, 10, 30\}$. In this way, we have

$$\frac{\sigma(n')}{n'} < e^\gamma \times \log \log n'$$

and therefore, it is enough to prove

$$\frac{\pi^2}{6} \times e^\gamma \times \log \log n' \leq e^\gamma \times \log \log n$$

which is the same as

$$\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$$

and thus, the proof is completed. ■

6 A Better Upper Bound

Lemma 6.1 For $x \geq 11$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - 0.12$$

where $q \leq x$ means all the primes lesser than or equal to x .

Proof For $x > 1$, we have

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + B + \frac{1}{\log^2 x}$$

where

$$B = 0.2614972128 \dots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [5]. This is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(C - \frac{1}{\log^2 x}\right)$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $\left(C - \frac{1}{\log^2 x}\right)$, then this complies with

$$\left(C - \frac{1}{\log^2 x}\right) > \left(0.31 - \frac{1}{\log^2 11}\right) > 0.12$$

for $x \geq 11$ and thus, we finally prove

$$\sum_{q \leq x} \frac{1}{q} < \log \log x + \gamma - \left(C - \frac{1}{\log^2 x}\right) < \log \log x + \gamma - 0.12.$$

■

7 On a Square Free Number

Theorem 7.1 *Given a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that q_1, q_2, \dots, q_m are odd prime numbers, the greatest prime divisor of n is greater than 7 and $3 \nmid n$, then we obtain the following inequality

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \log \log(2^{19} \times n).$$

Proof This proof is very similar with the demonstration in theorem 1.1 from the article reference [1]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [1]. Put $\omega(n) = m$ [1]. We need to prove the assertion for those integers with $m = 1$. From a square free number n , we obtain

$$(7.1) \quad \sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1)$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [1]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$(7.2) \quad \frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{q_i}\right) \leq e^\gamma \times \log \log(2^{19} \times q_i).$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{11}\right) \leq e^\gamma \times \log \log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$\left(1 + \frac{1}{q_i}\right) < \left(1 + \frac{1}{11}\right)$$

and

$$\log \log(2^{19} \times 11) < \log \log(2^{19} \times q_i)$$

which clearly implies that the inequality (7.2) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [1]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\begin{aligned} & \frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq \\ & e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \end{aligned}$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^\gamma \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq e^\gamma \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^\gamma \times n \times \log \log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

From the reference [1], we have if $0 < a < b$, then

$$(7.3) \quad \frac{\log b - \log a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}.$$

We can apply the inequality (7.3) to the previous one just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\begin{aligned} \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) &= \\ \log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} &= \log q_m. \end{aligned}$$

In this way, we obtain

$$\frac{q_m \times (\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log \log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} > \frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log \log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [1].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \log \log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^\gamma \times \log \log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log\left(\frac{\pi^2}{5.32}\right) + (\log(3+1) - \log 3) + \sum_{i=1}^m (\log(q_i+1) - \log q_i) \leq \gamma + \log \log \log(2^{19} \times n).$$

From the reference [1], we note

$$\log(q_1 + 1) - \log q_1 = \int_{q_1}^{q_1+1} \frac{dt}{t} < \frac{1}{q_1}.$$

In addition, note $\log\left(\frac{\pi^2}{5.32}\right) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log \log q_m < \gamma + \log \log \log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$ and therefore, it is enough to prove

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m$$

where $q_m \geq 11$. In this way, we only need to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log \log q_m - 0.12$$

which is true according to the lemma 6.1 when $q_m \geq 11$. In this way, we finally show the theorem is indeed satisfied. ■

8 Robin's Divisibility

Theorem 8.1 *Robins(n) holds for all $n > 5040$ when $3 \nmid n$. More precisely: every possible counterexample $n > 5040$ of the Robin's inequality must comply with $(2^{20} \times 3^{13}) \mid n$.*

Proof We will check the Robin's inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that q_1, q_2, \dots, q_m are prime numbers, a_1, a_2, \dots, a_m are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove

$$\frac{\sigma(n)}{n} < e^\gamma \times \log \log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{q_i + 1}{q_i} \leq e^\gamma \times \log \log n$$

according to the lemma 3.1. Using the formula (7.1), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^\gamma \times \log \log n$$

where $n' = q_1 \times \cdots \times q_m$ is the $\text{core}(n)$ [1]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [1]. Hence, we only need to prove the Robin's inequality is true when $2 \mid n'$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $2^k \mid n$ and $2^{20} \nmid n$ for some integer $1 \leq k \leq 19$ [3]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}) \leq e^\gamma \times n' \times \log \log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \log \log(2^{19} \times \frac{n'}{2}).$$

According to the formula (7.1) and $2 \mid n'$, we have

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \log \log(2^{19} \times \frac{n'}{2})$$

that is true according to the theorem 7.1 when $3 \nmid \frac{n'}{2}$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [3]. Consequently, we only need to prove the Robin's inequality is true for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. ■

Theorem 8.2 Robins(n) holds for all $n > 5040$ when $5 \nmid n$ or $7 \nmid n$.

Proof We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [6]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [6]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

Now, consider

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [6]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. However, we know the Robin's inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$ for some integer $1 \leq k \leq 12$ [3]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 5 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3 \times 7 \times m) < e^\gamma \times \log \log(2^a \times 3^b \times m)$$

when $b \geq 13$. ■

Theorem 8.3 *Robins(n) holds for all $n > 5040$ when a prime number $q_m \nmid n$ for $11 \leq q_m \leq 47$.*

Proof We know the Robin's inequality is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q_m \nmid m$ and $11 \leq q_m \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [6]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [6]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [6]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q_m) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $11 \leq q_m \leq 47$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 7 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$,

since this is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. Hence, we would have

$$f(2^a \times 3^b \times 7 \times q_m \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 7 \times q_m \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 7^c \times m)$$

when $c \geq 7$ and $11 \leq q_m \leq 47$. ■

Theorem 8.4 *Robins(n) holds for all $n > 5040$ when a prime number $q_m \nmid n$ for $53 \leq q_m \leq 113$.*

Proof We know the Robin's inequality is true for every natural number $n > 5040$ such that $11^k \mid n$ and $11^6 \nmid n$ for some integer $1 \leq k \leq 5$ [3]. We need to prove

$$f(n) < e^\gamma \times \log \log n$$

when $(2^{20} \times 3^{13} \times 11^6) \mid n$. Suppose that $n = 2^a \times 3^b \times 11^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 6$, $2 \nmid m$, $3 \nmid m$, $11 \nmid m$, $q_m \nmid m$ and $53 \leq q_m \leq 113$. Therefore, we need to prove

$$f(2^a \times 3^b \times 11^c \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 11^c \times m).$$

We know

$$f(2^a \times 3^b \times 11^c \times m) = f(11^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [6]. In addition, we know $f(11^c) < \frac{11}{10}$ for every natural number c [6]. In this way, we have

$$f(11^c) \times f(2^a \times 3^b \times m) < \frac{11}{10} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{121}{120} \times f(11) \times f(2^a \times 3^b \times m) = \frac{121}{120} \times f(2^a \times 3^b \times 11 \times m)$$

where $f(11) = \frac{12}{11}$ since f is multiplicative [6]. In addition, we know

$$\frac{121}{120} \times f(2^a \times 3^b \times 11 \times m) < f(q_m) \times f(2^a \times 3^b \times 11 \times m) = f(2^a \times 3^b \times 11 \times q_m \times m)$$

where $q_m \nmid m$, $f(q_m) = \frac{q_m+1}{q_m}$ and $53 \leq q_m \leq 113$. Nevertheless, we know the Robin's inequality is true for $2^a \times 3^b \times 11 \times q_m \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ such that $11^k \mid n$ and $11^6 \nmid n$ for some integer $1 \leq k \leq 5$ [3]. Hence, we would have

$$f(2^a \times 3^b \times 11 \times q_m \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 11 \times q_m \times m) < e^\gamma \times \log \log(2^a \times 3^b \times 11^c \times m)$$

when $c \geq 6$ and $53 \leq q_m \leq 113$. ■

9 Proof of Main Theorems

Theorem 9.1 *Robins(n) holds for all $n > 5040$ when a prime number $q_m \nmid n$ for $q_m \leq 113$.*

Proof This is a compendium of the results from the Theorems 8.1, 8.2, 8.3 and 8.4. ■

Theorem 9.2 Let $n > 5040$ and $n = r \times q_m$, where $q_m \geq 113$ denotes the largest prime factor of n . We have checked by computer, if $\text{Lagarias}(r)$ holds, then $\text{Lagarias}(n)$ holds.

Proof We need to prove

$$\sigma(n) \leq H_n + \exp(H_n) \times \log H_n.$$

We have that

$$\sigma(r) \leq H_r + \exp(H_r) \times \log H_r$$

since $\text{Lagarias}(r)$ holds. If we multiply by $(q_m + 1)$ the both sides of the previous inequality, then we obtain that

$$\sigma(r) \times (q_m + 1) \leq (q_m + 1) \times H_r + (q_m + 1) \times \exp(H_r) \times \log H_r.$$

We know that σ is submultiplicative (that is $\sigma(n) = \sigma(q_m \times r) \leq \sigma(q_m) \times \sigma(r)$) [1]. Moreover, we know that $\sigma(q_m) = (q_m + 1)$. In this way, we obtain that

$$\sigma(n) = \sigma(q_m \times r) \leq (q_m + 1) \times H_r + (q_m + 1) \times \exp(H_r) \times \log H_r.$$

Hence, it is enough to prove that

$$\begin{aligned} & (q_m + 1) \times H_r + (q_m + 1) \times \exp(H_r) \times \log H_r \\ & \leq H_n + \exp(H_n) \times \log H_n \\ & = H_{q_m \times r} + \exp(H_{q_m \times r}) \times \log H_{q_m \times r}. \end{aligned}$$

If we apply the lemma 2.3 to the previous inequality, then we could only need to analyze that

$$\begin{aligned} & (q_m + 1) \times \log(e^\gamma \times (r + 1)) + (q_m + 1) \times e^\gamma \times (r + 1) \times \log \log(e^\gamma \times (r + 1)) \\ & \leq \log(e^\gamma \times q_m \times r) + e^\gamma \times q_m \times r \times \log \log(e^\gamma \times q_m \times r). \end{aligned}$$

This has been checked by computer when the prime q_m is the largest prime factor of n and complies with $q_m \geq 113$. Indeed, we note by computer that the behavior of the subtraction between the both sides of this previous inequality is monotonically increasing as much as q_m and r become larger. ■

References

- [1] YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux*, 19(2):357–372, 2007. doi:10.5802/jtnb.591.
- [2] Harold M. Edwards. *Riemann's Zeta Function*. Dover Publications, 2001.
- [3] Alexander Hertlein. Robin's Inequality for New Families of Integers. *Integers*, 18, 2018.
- [4] Jeffrey C. Lagarias. An Elementary Problem Equivalent to the Riemann Hypothesis. *The American Mathematical Monthly*, 109(6):534–543, 2002. doi:10.2307/2695443.
- [5] J. Barkley Rosser and Lowell Schoenfeld. Approximate Formulas for Some Functions of Prime Numbers. *Illinois Journal of Mathematics*, 6(1):64–94, 1962. doi:10.1215/ijm/1255631807.
- [6] Robert Vojak. On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds. *arXiv preprint arXiv:2005.09307*, 2020.

CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
Email: vega.frank@gmail.com