



## Deep on Goldbach's Conjecture

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## Deep on Goldbach's conjecture

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### Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. The Goldbach's conjecture has been verified for every even number  $N \leq 4 \cdot 10^{18}$ . In this note, we prove that for every even number  $N \geq 4 \cdot 10^{18}$ , if there is a prime  $p$  and a natural number  $m$  such that  $n < p < N - 1$ ,  $p + m = N$ ,  $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$  and  $p$  is coprime with  $m$ , then  $m$  is necessarily a prime number when  $N = 2 \cdot n$  and  $\sigma(m)$  is the sum-of-divisors function of  $m$ . The previous inequality  $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$  holds whenever  $\frac{N}{e^{\gamma \cdot m \cdot \log \log m}} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$  also holds and  $m \geq 11$  is an odd number, where  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant and  $\log$  is the natural logarithm. This implies that the Goldbach's conjecture is true when the Riemann hypothesis is true.

**Keywords:** Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function

**MSC Classification:** 11A41 , 11A25

## 1 Introduction

As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$

$$\sum_{d|n} d,$$

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where  $d \mid n$  means the integer  $d$  divides  $n$ . Define  $s(n)$  as  $\frac{\sigma(n)}{n}$ . In number theory, the  $p$ -adic order of an integer  $n$  is the exponent of the highest power of the prime number  $p$  that divides  $n$ . It is denoted  $\nu_p(n)$ . Equivalently,  $\nu_p(n)$  is the exponent to which  $p$  appears in the prime factorization of  $n$ . We can state the sum-of-divisors function of  $n$  as

$$\sigma(n) = \prod_{p \mid n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers  $p$  which divide  $n$ . In addition, the well-known Euler's totient function  $\varphi(n)$  can be formulated as

$$\varphi(n) = n \cdot \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

Chen's theorem states that every sufficiently large even number can be written as the sum of either two primes, or a prime and a semiprime (the product of two primes) [1]. Tomohiro Yamada using an explicit version of Chen's theorem showed that every even number greater than  $e^{e^{36}} \approx 1.7 \cdot 10^{1872344071119343}$  is the sum of a prime and a product of at most two primes [2]. A natural number is called  $k$ -almost prime if it has  $k$  prime factors [3]. A natural number is prime if and only if it is 1-almost prime, and semiprime if and only if it is 2-almost prime. Let  $N$  be a sufficiently large even integer. Ying Chun Cai proved that the equation

$$N = p + P_2, \quad p \leq N^{0.95},$$

is solvable, where  $p$  denotes a prime and  $P_2$  denotes an almost prime with at most two prime factors [3]. The Goldbach's conjecture has been verified for every even number  $N \leq 4 \cdot 10^{18}$  [4]. In mathematics, two integers  $a$  and  $b$  are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem.

**Theorem 1** *For every even number  $N \geq 4 \cdot 10^{18}$ , if there is a prime  $p$  and a natural number  $m$  such that  $n < p < N - 1$ ,  $p + m = N$ ,  $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$  and  $p$  is coprime with  $m$ , then  $m$  is necessarily a prime number when  $N = 2 \cdot n$ . The previous inequality  $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$  holds whenever  $\frac{N}{e^{\gamma \cdot m \cdot \log \log m}} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$  also holds and  $m \geq 11$  is an odd number, where  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant and  $\log$  is the natural logarithm. This implies that the Goldbach's conjecture is true when the Riemann hypothesis is true.*

## 2 Proof of Theorem 1

*Proof* Suppose that there is an even number  $N \geq 4 \cdot 10^{18}$  which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers  $(n - k, n + k)$

where  $n = \frac{N}{2}$ ,  $k < n - 1$  is a natural number,  $n + k$  and  $n - k$  are coprime integers and  $n + k$  is prime. By definition of the functions  $\sigma(x)$  and  $\varphi(x)$ , we know that

$$2 \cdot N = \sigma((n - k) \cdot (n + k)) - \varphi((n - k) \cdot (n + k))$$

when  $n - k$  is also prime. We notice that

$$2 \cdot N < \sigma((n - k) \cdot (n + k)) - \varphi((n - k) \cdot (n + k))$$

when  $n - k$  is not a prime. Certainly, we see that  $(n - k) + (n + k) = N$  and thus, the inequality

$$2 \cdot ((n - k) + (n + k)) + \varphi((n - k) \cdot (n + k)) < \sigma((n - k) \cdot (n + k))$$

holds when  $n - k$  is not a prime. That is equivalent to

$$2 \cdot ((n - k) + (n + k)) + \varphi(n - k) \cdot \varphi(n + k) < \sigma(n - k) \cdot \sigma(n + k)$$

since the functions  $\sigma(x)$  and  $\varphi(x)$  are multiplicative. Let's divide both sides by  $(n - k) \cdot (n + k)$  to obtain that

$$2 \cdot \left( \frac{(n - k) + (n + k)}{(n - k) \cdot (n + k)} \right) + \frac{\varphi(n - k)}{n - k} \cdot \frac{\varphi(n + k)}{n + k} < s(n - k) \cdot s(n + k).$$

We know that

$$s(n - k) \cdot s(n + k) > 1$$

since  $s(m) > 1$  for every natural number  $m > 1$  [5]. Moreover, we could see that

$$2 \cdot \left( \frac{(n - k) + (n + k)}{(n - k) \cdot (n + k)} \right) = \frac{2}{n + k} + \frac{2}{n - k}$$

and therefore,

$$1 > \frac{2}{n + k} + \frac{2}{n - k} + \frac{\varphi(n - k)}{n - k} \cdot \frac{\varphi(n + k)}{n + k}.$$

It is enough to see that

$$1 > \frac{2}{2 \cdot 10^{18}} + \frac{2}{9} + \frac{2}{3} \geq \frac{2}{n + k} + \frac{2}{n - k} + \frac{\varphi(n - k)}{n - k} \cdot \frac{\varphi(n + k)}{n + k}$$

when  $n + k$  is prime and  $n - k$  is composite for  $N \geq 4 \cdot 10^{18}$ . Indeed, when  $n + k$  is prime and  $n - k$  is composite, then  $n + k > 2 \cdot 10^{18}$  and  $n - k \geq 9$  for  $N \geq 4 \cdot 10^{18}$ . Under our assumption, all these pairs of positive integers  $(n - k, n + k)$  imply that

$$2 \cdot N < \sigma((n - k) \cdot (n + k)) - \varphi((n - k) \cdot (n + k))$$

holds whenever  $n = \frac{N}{2}$ ,  $k < n - 1$  is a natural number,  $n + k$  and  $n - k$  are coprime integers and  $n + k$  is prime. Hence, we have

$$N < \frac{1}{2} \cdot (\sigma(n - k) \cdot \sigma(n + k) - \varphi(n - k) \cdot \varphi(n + k)).$$

Since  $n + k$  is prime, then

$$\begin{aligned} \frac{\varphi(n + k)}{1 + n^{0.889}} &= \frac{n + k - 1}{1 + n^{0.889}} \\ &\geq \frac{n}{1 + n^{0.889}} \\ &\geq 2 \cdot \left( e^\gamma \cdot \log \log(n - 1) + \frac{2.5}{\log \log(n - 1)} \right)^2 \\ &\geq 2 \cdot \left( e^\gamma \cdot \log \log(n - k) + \frac{2.5}{\log \log(n - k)} \right)^2 \end{aligned}$$

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$$\begin{aligned}
 &> 2 \cdot \left( \frac{n-k}{\varphi(n-k)} \right)^2 \\
 &= \frac{n-k}{\varphi(n-k)} \cdot 2 \cdot \prod_{q|(n-k)} \left( \frac{q}{q-1} \right) \\
 &> s(n-k) \cdot 2 \cdot \prod_{q|(n-k)} \left( \frac{q}{q-1} \right) \\
 &= \frac{2 \cdot \sigma(n-k)}{(n-k) \cdot \prod_{q|(n-k)} \left( 1 - \frac{1}{q} \right)} \\
 &= \frac{2 \cdot \sigma(n-k)}{\varphi(n-k)}
 \end{aligned}$$

when we know that  $\frac{b}{\varphi(b)} < e^\gamma \cdot \log \log(b) + \frac{2.5}{\log \log(b)}$  holds for every odd number  $b \geq 3$  [6]. Moreover, we have

$$\frac{n}{1+n^{0.889}} \geq 2 \cdot \left( e^\gamma \cdot \log \log(n-1) + \frac{2.5}{\log \log(n-1)} \right)^2$$

for every natural number  $n \geq 2 \cdot 10^{18}$  under the supposition that  $N \geq 4 \cdot 10^{18}$ . Certainly, the function

$$f(x) = \frac{x}{1+x^{0.889}} - 2 \cdot \left( e^\gamma \cdot \log \log(x-1) + \frac{2.5}{\log \log(x-1)} \right)^2$$

is strictly increasing and positive for every real number  $x \geq 2 \cdot 10^{18}$  because of its derivative is greater than 0 for all  $x \geq 2 \cdot 10^{18}$  and it is positive in the value of  $2 \cdot 10^{18}$ . Furthermore, it is known that  $\prod_{q|b} \left( \frac{q}{q-1} \right) = \frac{b}{\varphi(b)} > s(b) = \frac{\sigma(b)}{b}$  for every natural number  $b \geq 2$  [5]. Finally, we would have that

$$-\frac{1}{2} \cdot \varphi(n-k) \cdot \varphi(n+k) < -\sigma(n-k) \cdot (1+n^{0.889})$$

and so,

$$N < \frac{1}{2} \cdot \sigma(n-k) \cdot \sigma(n+k) - \sigma(n-k) \cdot (1+n^{0.889}).$$

We would have

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 < \frac{\sigma(n+k)}{2}$$

which is

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} < n.$$

In this way, we obtain a contradiction when we assume that  $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ . By reductio ad absurdum, the natural number  $n-k$  is necessarily prime when  $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ . Moreover, we know that  $\sigma(b) < e^\gamma \cdot b \cdot \log \log b$  holds for every odd number  $b \geq 11$  [5]. Consequently, the inequality  $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$  holds whenever  $\frac{N}{e^\gamma \cdot (n-k) \cdot \log \log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$  also holds and  $(n-k) \geq 11$  is an odd number. In 2014, Dudek proved that the Riemann hypothesis implies that for all  $x \geq 2$  there is a prime  $p$  satisfying [7]

$$x - \frac{4}{\pi} \sqrt{x} \log x < p \leq x.$$

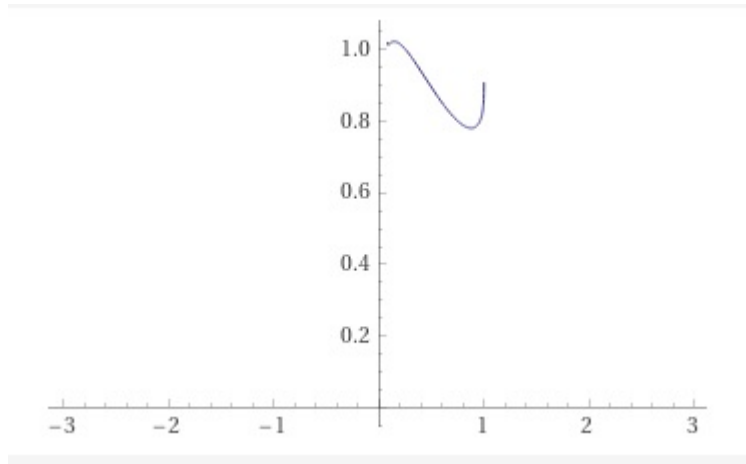


Fig. 1 Plot of function  $H_4(x)$  [8]

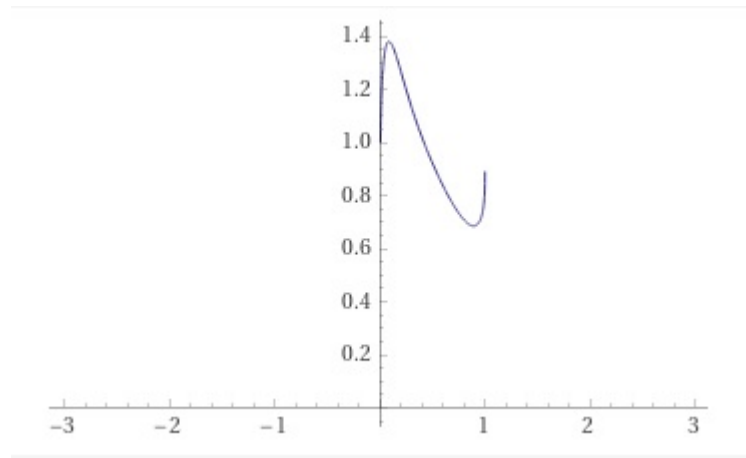


Fig. 2 Plot of function  $H_8(x)$  [9]

In this way, there is always a prime  $n + k$  for  $\frac{4}{\pi} \cdot \sqrt{n} \cdot \log n \leq k \leq \frac{8}{\pi} \cdot \sqrt{n} \cdot \log n$ . However, we know the inequality  $\frac{2 \cdot n}{e^{\gamma \cdot (n-k)} \cdot \log \log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$  holds for all positive integers  $n \geq 2 \cdot 10^{18}$  and  $\frac{4}{\pi} \cdot \sqrt{n} \cdot \log n \leq k \leq \frac{8}{\pi} \cdot \sqrt{n} \cdot \log n$  since the function  $H_a(x) = \frac{x}{(x - \frac{a}{\pi} \cdot \sqrt{x} \cdot \log x) \cdot \log \log(x - \frac{a}{\pi} \cdot \sqrt{x} \cdot \log x)} + x^{0.889} + 1 + \frac{x - \frac{a}{\pi} \cdot \sqrt{x} \cdot \log x - 1}{2} - x$  is positive for all  $x \geq 2 \cdot 10^{18}$  and  $a \in \{4, 8\}$  (See Figures 1 and 2). Certainly, we know that  $H_a(n) \leq \frac{2 \cdot n}{e^{\gamma \cdot (n-k)} \cdot \log \log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} - n$  for all positive integers  $n \geq 2 \cdot 10^{18}$  and  $\frac{4}{\pi} \cdot \sqrt{n} \cdot \log n \leq k \leq \frac{8}{\pi} \cdot \sqrt{n} \cdot \log n$ , where we select the appropriated value of  $4 \leq a \leq 8$  according to the value of  $k$ .  $\square$

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