

# Spectra of finitely presented lattice-ordered Abelian groups and MV-algebras, part 1.

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## 1 Introduction

This is the first part of a series of two abstract, the second one being by Daniel McNeill.

If  $X$  is any topological space, its collection of opens sets  $\mathcal{O}(X)$  is a complete distributive lattice, with joins given by unions, and meets given by  $\bigwedge_{U \in S} U := \text{int}(\bigcap_{U \in S} U)$ , for any  $S \subseteq \mathcal{O}(X)$ . Moreover,  $\mathcal{O}(X)$  is a *frame*, i.e. it satisfies the *frame law*

$$U \wedge \left( \bigvee_{V \in S} V \right) = \bigvee_{V \in S} (U \wedge V), \quad \text{for any } S \subseteq \mathcal{O}(X), U \in \mathcal{O}(X). \quad (1)$$

Thus,  $\mathcal{O}(X)$  is also a Heyting algebra whose implication is defined as

$$U \rightarrow V := \bigvee \{Z \mid U \wedge Z \subseteq V\}, \quad \text{for any } U, V \in \mathcal{O}(X). \quad (2)$$

(See e.g. [6] for background.) When  $X$  is equipped with a distinguished basis  $D$  for its topology, closed under finite meets and joins, one can investigate situations where  $D$  is also closed under the implication (2), i.e., where  $D$  is a Heyting subalgebra of  $\mathcal{O}(X)$ .

Recall that  $X$  is a *spectral space* if it is compact and  $T_0$ , its collection  $D$  of compact open subsets forms a basis which is closed under finite intersections and unions, and  $X$  is *sober*: any closed set that cannot be written as the union of two proper closed subsets, has a dense point. (In this case, the latter point is unique, because  $X$  is  $T_0$ .) By Stone duality, spectral spaces are precisely the spaces arising as sets of prime ideals of some distributive lattice, topologised with the Stone or hull-kernel topology. Specifically, given such a spectral space  $X$ , its collection of compact open sets  $D$  is (naturally isomorphic to) the distributive lattice dual to  $X$  under Stone duality. We are going to exhibit a significant class of such spaces for which  $D$  is a Heyting subalgebra of  $\mathcal{O}(X)$ . We work with lattice-ordered Abelian groups and vector spaces. Using Mundici’s  $\Gamma$ -functor [8] the results can be rephrased in terms of *MV-algebras*, the algebraic semantics of Lukasiewicz infinite-valued propositional logic [5].

Recall that a *lattice-ordered Abelian group* [4], or  *$\ell$ -group* for short, is an Abelian group which is also a lattice, and is such that the group operation distributes over both meets and joins. Similarly, a *vector lattice* (also known as a *Riesz space*), is a lattice-ordered real vector

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space such that addition distributes over meets and joins, and multiplication by non-negative real scalars also distributes over meets and joins. A  $\mathbb{Q}$ -vector lattice is defined analogously replacing real vector spaces with rational vector spaces. The classes of  $\ell$ -groups, vector lattices, and  $\mathbb{Q}$ -vector lattices are varieties of algebras. Hence free objects in each class exist, and finitely presented objects are standardly defined as quotients of finitely generated free objects modulo a finitely generated congruence. Further, since each structure in question has an obvious distributive-lattice reduct, each of the three varieties is congruence-distributive. It is known that the complete distributive algebraic lattice  $\text{Con } G$  of the congruences<sup>1</sup> of an  $\ell$ -group (or vector lattice, or  $\mathbb{Q}$ -vector lattice)  $G$  is in fact relatively pseudo-complemented, i.e. is a Heyting algebra. We write  $\mathbf{K}(G)$  for the subset of  $\text{Con } G$  consisting of the finitely generated congruences of  $G$ . It can be shown that when  $G$  is finitely presented, then  $\mathbf{K}(G)$  is a sublattice of  $\text{Con } G$ ; see Section 2 for more details on this.

**Theorem 1.1.** *Let  $G$  be a finitely presented  $\ell$ -group (or vector lattice, or  $\mathbb{Q}$ -vector lattice), and let  $\text{Con } G$  be the Heyting algebra consisting of the congruences on  $G$ . Then the congruence*

$$\theta \rightarrow \eta := \bigvee \{ \zeta \mid \theta \wedge \zeta \subseteq \eta \} \quad (3)$$

*is finitely generated whenever  $\theta, \eta \in \mathbf{K}(G)$ . Equivalently,  $\mathbf{K}(G)$  is a Heyting subalgebra of  $\text{Con } G$ .*

## 2 The Spectral Space of $G$

Let  $G$  be an  $\ell$ -group (or a vector lattice, or a  $\mathbb{Q}$ -vector lattice). A congruence  $\theta$  on  $G$  is *proper* if  $\theta \neq G \times G$ , *prime* if the quotient  $G/\theta$  is totally ordered, and *maximal* if  $\theta$  is proper and whenever  $\eta \supseteq \theta$  is a congruence on  $G$ , then either  $\eta = \theta$  or  $\eta = G \times G$ .

Let  $\text{Spec } G$  be the set of all prime congruences of  $G$ . We define two maps  $\mathbb{V}: 2^G \rightarrow 2^{\text{Spec } G}$  and  $\mathbb{I}: 2^{\text{Spec } G} \rightarrow 2^G$  by setting, for each  $S \subseteq G$  and for each  $E \subseteq \text{Spec } G$ ,

$$\mathbb{V}(S) := \{ \mathfrak{p} \in \text{Spec } G \mid (s, 0) \in \mathfrak{p} \text{ for all } s \in S \} \text{ and } \mathbb{I}(E) := \{ g \in G \mid (g, 0) \in \mathfrak{p} \text{ for all } \mathfrak{p} \in E \}.$$

The pair  $(\mathbb{I}, \mathbb{V})$  forms a (*contravariant*) *Galois connection*: for all  $S \subseteq G$  and  $E \subseteq \text{Spec } G$ ,  $S \subseteq \mathbb{I}(E)$  if and only if  $E \subseteq \mathbb{V}(S)$ . This ensures that the maps  $\mathbb{I} \circ \mathbb{V}: 2^G \rightarrow 2^G$  and  $\mathbb{V} \circ \mathbb{I}: 2^{\text{Spec } G} \rightarrow 2^{\text{Spec } G}$  are closure operators. In particular,  $\mathbb{I} \circ \mathbb{V}: 2^G \rightarrow 2^G$  is *algebraic*, i.e. it satisfies  $\mathbb{I} \circ \mathbb{V}(S) = \bigcup \mathbb{I} \circ \mathbb{V}(F)$ , as  $F$  ranges over the finite subsets of  $S$ . Moreover, it can be shown that  $\mathbb{V} \circ \mathbb{I}: 2^{\text{Spec } G} \rightarrow 2^{\text{Spec } G}$  is *topological*, i.e. it commutes with finite unions. Thus, the operator  $\mathbb{V} \circ \mathbb{I}$  gives rise to a topology (the *spectral* or *hull-kernel topology*) on  $\text{Spec } G$ , whose closed sets are exactly the subsets  $E \subseteq \text{Spec } G$  such that  $\mathbb{V} \circ \mathbb{I}(E) = E$ . We will shortly see that  $\text{Spec } G$  is a spectral space; it is easy to show directly that it is  $T_0$ , whence it carries a specialization order. Specifically, the *specialization order*  $\preceq$  on  $\text{Spec } G$  defined by

$$\mathfrak{p} \preceq \mathfrak{q} \text{ if, and only if, } \mathfrak{p} \in \mathbb{V} \circ \mathbb{I}(\mathfrak{q})$$

coincides with the set-theoretic inclusion  $\subseteq$ . It is well known [4, 2.4.1 and 10.1.11] that  $(\text{Spec } G, \preceq)$  is a *root system*: each *upper set*  $\uparrow \mathfrak{p} := \{ \mathfrak{q} \in \text{Spec } G \mid \mathfrak{p} \preceq \mathfrak{q} \}$  is linearly ordered. This property is usually called the *complete normality* of  $\text{Spec } G$ . It is also known [4, 10.2.2]

<sup>1</sup>We recall that congruences of  $\ell$ -groups (or vector lattices, or  $\mathbb{Q}$ -vector lattices) are in one-one inclusion-preserving correspondence with kernels of homomorphisms, known as  *$\ell$ -ideals*. These can be characterised as the order-convex subalgebras in each of the varieties under consideration.

that  $\text{Spec } G$  is a compact space if, and only if,  $G$  may be equipped with a (*strong order*) *unit*, i.e. there exists  $u \in G$  such that for all  $g \in G$  we have  $nu \geq g$  for some integer  $n \geq 1$ . In this case  $G$  is called a *unital  $\ell$ -group* (or vector lattice, or  $\mathbb{Q}$ -vector lattice). For the remainder of this section we always assume that  $G$  is unital.

It turns out that  $\text{Spec } G$  may be regarded as the spectrum of prime ideals of a distributive lattice. Finitely generated congruences on  $G$  are the same thing as *principal* (i.e. singly generated) congruences on  $G$ . Indeed, writing  $\langle T \rangle$  for the congruence on  $G$  generated by  $T \subseteq G \times G$ , we have  $\langle (g, 0) \rangle = \langle (|g|, 0) \rangle$ , where  $|g| := g \vee -g$  is the *absolute value* of  $g \in G$ ; and, moreover,  $\langle (g, 0) \rangle \cap \langle (h, 0) \rangle = \langle (|g| \wedge |h|, 0) \rangle$  and  $\langle (g, 0) \rangle \vee \langle (h, 0) \rangle = \langle (|g| \vee |h|, 0) \rangle$  for all  $g, h \in G$ . Observe that the top element of  $\text{Con } G$ , namely, the improper congruence  $G \times G$ , is principal, because  $\langle (u, 0) \rangle = G \times G$  for any unit  $u \in G$ . It follows that  $K(G)$  is a (distributive) sublattice of  $\text{Con } G$ . Writing  $\text{Spec}_{\text{DL}} K(G)$  for the spectrum of prime ideals of  $K(G)$  with the Stone topology, it can now be shown that  $\text{Spec}_{\text{DL}} K(G)$  is homeomorphic to  $\text{Spec } G$ ; the homeomorphism takes a prime ideal  $P \subseteq K(G)$  to the congruence  $\bigcap P$  on  $G$ .

By the foregoing, together with Stone duality, we have that  $\text{Spec } G$  equipped with the spectral topology is a spectral space. The correspondence

$$\langle (g, 0) \rangle \in K(G) \quad \longmapsto \quad \text{Spec } G \setminus \mathbb{V}(g) \tag{4}$$

yields a lattice homomorphism between  $K(G)$  and the lattice of compact open subsets of  $\text{Spec } G$ .

### 3 Finitely Presented Structures, and Polyhedra

For background on polyhedral geometry see [9]. A subset  $C$  of finite-dimensional Euclidean space  $\mathbb{R}^n$  is *convex* if it contains, together with  $p, q \in C$ , the entire line segment  $\{\lambda p + (1 - \lambda)q \mid \lambda \in \mathbb{R}, 0 \leq \lambda \leq 1\}$ . Given a subset  $S \subseteq \mathbb{R}^n$  of finite-dimensional Euclidean space, its *convex hull* of  $S$ , written  $\text{conv } S$ , is the intersection of all convex subsets of  $\mathbb{R}^n$  that contain it. A *polytope* in  $\mathbb{R}^n$  is the convex hull of a finite subset of  $\mathbb{R}^n$ . Polytopes are thus compact and convex. A *polyhedron* in  $\mathbb{R}^n$  is any subset that can be written as the union of finitely many polytopes. Polyhedra are thus compact, but not necessarily convex. A polytope is *rational* if it can be written as the convex hull of a finite subset of  $\mathbb{Q}^n \subseteq \mathbb{R}^n$ . A polyhedron is *rational* if it can be written as the union of finitely many rational polytopes.

For  $P \subseteq \mathbb{R}^n$  a polyhedron, a continuous map  $f: P \rightarrow \mathbb{R}$  is *piecewise linear* (*P.L.* for short) if there exist finitely many affine linear functions  $l_1, \dots, l_u: \mathbb{R}^n \rightarrow \mathbb{R}$  such that, for each  $p \in P$ , we have  $f(p) = l_{i_p}(p)$  for some  $i_p \in \{1, \dots, u\}$ . If the  $l_i$ 's in this definition can be chosen to have rational coefficients (i.e. each  $l_i$  is the extension to  $\mathbb{R}^n$  of an affine linear map  $\mathbb{Q}^n \rightarrow \mathbb{Q}$ ) then  $f$  is a *rational P.L. map*. And if they can be chosen to have integer coefficients (i.e. each  $l_i$  is the extension to  $\mathbb{R}^n$  of a  $\mathbb{Z}$ -module map  $\mathbb{Z}^n \rightarrow \mathbb{Z}$ ) then  $f$  is a  *$\mathbb{Z}$ -map*.

**Definition 3.1.** For any polyhedron  $P \subseteq \mathbb{R}^n$ , we write  $\nabla(P)$  to denote the collection of all P.L. maps  $P \rightarrow \mathbb{R}$ . Assume further that  $P$  is rational. Then we write  $\nabla_{\mathbb{Q}}(P)$  the collections of all rational P.L. maps  $P \rightarrow \mathbb{R}$ , and  $\nabla_{\mathbb{Z}}(P)$  for the collection of all  $\mathbb{Z}$ -maps  $P \rightarrow \mathbb{R}$ .

Write  $1_P: P \rightarrow \mathbb{R}$  for the function constantly equal to 1 on  $P$ . It is an exercise to prove that:

- $\nabla_{\mathbb{Z}}(P)$  is an  $\ell$ -group having  $1_P$  as a unit.
- $\nabla_{\mathbb{Q}}(P)$  is a  $\mathbb{Q}$ -vector lattice having  $1_P$  as a unit.
- $\nabla(P)$  is a vector lattice having  $1_P$  as a unit.

**Lemma 3.2.** *Let  $G$ ,  $R$ , and  $V$  be a finitely presented  $\ell$ -group,  $\mathbb{Q}$ -vector lattice, and vector lattice, respectively. Let  $u$  be a unit of  $G$ ,  $R$ , or  $V$ , respectively. Then there exist rational polyhedra  $P$  and  $Q$ , and a polyhedron  $T$ , such that:*

- $(G, u) \cong_u (\nabla_{\mathbb{Z}}(P), 1_P)$ .
- $(R, u) \cong_u (\nabla_{\mathbb{Q}}(Q), 1_Q)$ .
- $(V, u) \cong_u (\nabla(T), 1_T)$ .

Moreover,  $P$ ,  $Q$ , and  $T$ , regarded as topological spaces, are naturally homeomorphic to  $\text{Max } G$ ,  $\text{Max } R$ , and  $\text{Max } V$ , respectively.

**Remark 3.3.** The representation theorem given in the preceding lemma is part of *Baker-Beynon duality* and its variants. See [1, 2, 3]. For MV-algebras see [7].

## 4 The Heyting Algebra of Principal Congruences of Finitely Presented Structures

To prove Theorem 1.1 we relate the spectral space of finitely presented structures with the geometric representation recalled in Section 3. Given a polyhedron  $P$ , let  $\text{Sub } P$  denote the collection of all polyhedra contained in  $P$ . Observe that  $\text{Sub } P$  is a distributive lattice under intersections and unions, with top element  $P$  and bottom element  $\emptyset$ . If  $P$  is rational, we let  $\text{Sub}_{\mathbb{Q}} P$  denote the sublattice of  $\text{Sub } P$  consisting of the rational polyhedra contained in  $P$ . Given a lattice  $L$  we write  $L^{\text{op}}$  to denote the order-dual of  $L$  obtained by reversing the order of  $L$ .

**Lemma 4.1.** (i) *Let  $(G, u)$  be a finitely presented  $\ell$ -group with unit  $u$ , and let  $P$  be a rational polyhedron such that there is a unital isomorphism  $\varphi: (G, u) \rightarrow (\nabla_{\mathbb{Z}}(P), 1_P)$  as in Lemma 3.2.*

$$\langle (g, 0) \rangle \in \mathbf{K}(G) \quad \longmapsto \quad \varphi(g)^{-1}(0) \in \text{Sub}_{\mathbb{Q}} P \quad (5)$$

*yields a lattice isomorphism between  $\mathbf{K}(G)$  and  $(\text{Sub}_{\mathbb{Q}} P)^{\text{op}}$ . Similarly, if  $(R, u)$  is a finitely presented  $\mathbb{Q}$ -vector lattice, there is a lattice isomorphism between  $\mathbf{K}(R)$  and  $(\text{Sub}_{\mathbb{Q}} Q)^{\text{op}}$ , where  $Q$  is a rational polyhedron as in Lemma 3.2.*

(ii) *Let  $(V, u)$  be a finitely presented vector lattice with unit  $u$ . let  $T$  be a polyhedron such that there is a unital isomorphism  $\varphi: (V, u) \rightarrow (\nabla(T), 1_T)$  as in Lemma 3.2.*

$$\langle (v, 0) \rangle \in \mathbf{K}(V) \quad \longmapsto \quad \varphi(v)^{-1}(0) \in \text{Sub } T \quad (6)$$

*yields a lattice isomorphism between  $\mathbf{K}(V)$  and  $(\text{Sub } T)^{\text{op}}$ .*

**Remark 4.2.** Observe that, as a consequence of the preceding lemma together with Stone duality, if  $P$  is a rational polyhedron then  $\text{Spec } \nabla_{\mathbb{Q}}(P)$  is homeomorphic to  $\text{Spec } \nabla_{\mathbb{Z}}(P)$ , while  $\text{Spec } \nabla_{\mathbb{Q}}(P)$  is not homeomorphic to  $\text{Spec } \nabla(P)$ .

The proof of Theorem 1.1 can now be reduced to the following geometric lemma.

**Lemma 4.3.** *Let  $A$  and  $B$  be two polyhedra in some Euclidean space  $\mathbb{R}^n$ . Then the topological closure of  $A \setminus B$  is a polyhedron of  $\mathbb{R}^n$ . Moreover, if  $A$  and  $B$  are rational polyhedra, then the topological closure of  $A \setminus B$  is a rational polyhedron, too.*

The preceding lemma asserts that the lattice  $\text{Sub } T$  of a polyhedron (or the lattice  $\text{Sub}_{\mathbb{Q}} P$  of a rational polyhedron) has the structure of a dual Heyting algebra. Via Lemma 4.1 this translates to the fact that the lattice  $\mathbf{K}(G)$  of principal congruences of a finitely presented  $\ell$ -group (or vector lattice, or  $\mathbb{Q}$ -vector lattice)  $G$  is closed under the Heyting implication of  $\text{Con } G$ .

## 5 Compactness of Spaces of Minimal Primes

We give two consequences of our main result that will be used in the second part of this abstract. For  $L$  a distributive lattice, Speed [10] gave a necessary and sufficient condition for the subspace of minimal primes of  $\text{Spec}_{\text{DL}} L$  to be compact. It is elementary to verify that whenever  $\text{Spec}_{\text{DL}} L$  is such that its compact open subsets form a Heyting subalgebra of the Heyting algebra of open sets  $\mathcal{O}(\text{Spec}_{\text{DL}} L)$ ,  $L$  satisfies Speed's condition. The converse fails. Recalling that a compact Hausdorff space is *Boolean* if its clopen sets form a basis for the topology, we have:

**Corollary 5.1.** *Let  $G$  be a finitely presented unital  $\ell$ -group (or unital vector lattice, or unital  $\mathbb{Q}$ -vector lattice). Then the subspace  $\text{Min } G \subseteq \text{Spec } G$  of minimal prime congruences of  $G$  is a Boolean space.*

Let  $G$  be a finitely presented unital  $\ell$ -group (or unital vector lattice, or unital  $\mathbb{Q}$ -vector lattice), and let  $\mathfrak{m}$  be a maximal congruence of  $G$ . The *germinal congruence at  $\mathfrak{m}$*  is defined by

$$\text{germ } \mathfrak{m} := \bigcap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{p},$$

where  $\mathfrak{p}$  ranges over prime congruences of  $G$ . For each  $\mathfrak{p} \in \text{Spec } G$ , we denote by  $\downarrow \mathfrak{p}$  the *lower set*  $\{\mathfrak{q} \in \text{Spec } G \mid \mathfrak{q} \preceq \mathfrak{p}\}$ . Arguments similar to the ones proving the preceding corollary yield:

**Corollary 5.2.** *Let  $G$  be a finitely presented unital  $\ell$ -group (or unital vector lattice, or unital  $\mathbb{Q}$ -vector lattice). Let  $\mathfrak{m}$  be any maximal congruence of  $G$ . Then  $\downarrow \mathfrak{m} \subseteq \text{Spec } G$  equipped with the subspace topology is homeomorphic to  $\text{Spec}(G/\text{germ } \mathfrak{m})$ , and is therefore a spectral space. The subspace  $\text{Min}(\downarrow \mathfrak{m})$  of  $\downarrow \mathfrak{m}$  is a Boolean space.*

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